THE MATHEMATICS STUDENT

A Quarterly Dedicated to the Service of Students and Teachers of Mathematics in India

Edited by

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With the co-operation of

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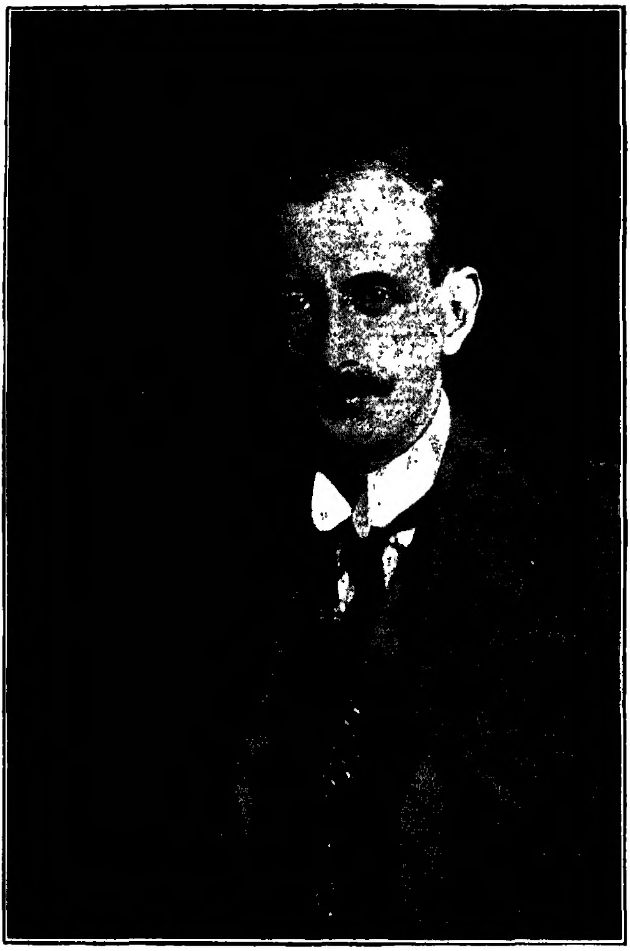

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VAGUE HISTORICAL VIEWS RELATING TO THE NEGATIVE NUMBERS.

By G. A. MILLER,

University of Illinois.

In the history of negative numbers it is desirable to distinguish sharply between the following three points: their correct use, their correct theory and their introduction as a permanent element of mathematics. It is well-known that the Hindus used these numbers correctly and that this is the earliest such use thereof that has thus far been definitely established. On the other hand, the eighteenth century European mathematicians did not yet generally possess a correct theory of these numbers so that J. Tropfke could truly say in the third edition of his noted *Geschichte der Elementar-Mathematik*, Volume 2 (1933), page 101, that the eighteenth century suffered because a generally satisfactory introduction of these numbers was then lacking. The symbols + and — were employed in calculations and were then used at a bound to represent positive and negative numbers.

The fact that even in our day there are eminent European scientists who do not fully comprehend the theory of negative numbers is established by the following quotation from the preface of a well-known recent book: "Personally I more and more come to regard the purely formal and mathematical presentation of physical theories as a disguise and evasion of the real problems rather than as a solution of them. I have tried in other fields to show the incredible confusions, of which the whole world is now one seething example, that have followed from the invention by the Hindu mathematicians of negative quantities, and their justification from their analogy to debt, so that naturally I am not among those

who can bow down and worship the square root of minus one." F. SODDY: *The Interpretation of the Atom*, 1932.

It should be emphasized in this connection that the use of negative numbers is not fully justified from their analogy to debt. The product of two debts is not a credit. When debts are multiplied, they are multiplied by positive numbers and not by negative numbers so as to become a credit. The history of negative numbers becomes clear only by noting that the analogy of negative numbers to debt relates only to some of the properties of these numbers while it is foreign to other properties thereof. Even such an eminent mathematician as H. Cardan tried to prove in the latter half of the sixteenth century that it is incorrect to say that $(-a).(-b) = +ab$, and in the noted algebra of C. Clavius (1608) the author ascribes to the weakness of the human mind the impossibility of understanding this rule, but he did not doubt its correctness in view of the fact that it had been verified by many examples.

The theory of negative numbers with respect to the operations of addition and subtraction is much more simple than their theory with respect to the operations of multiplication and division. In the former theory, the analogy of these numbers to debt when the corresponding positive numbers represent credit and to distances in the opposite direction from those represented by the corresponding positive numbers are illuminating. In trigonometry and in analytic geometry, negative numbers are especially useful with respect to the operations of addition and subtraction. Hence it is perhaps natural that some mathematical historians have unduly emphasized this analogy in presenting the extension of the number concept so as to include negative numbers. This extension, however, implied also a correct theory of these numbers with respect to the operations of multiplication and division, and the latter theory does not seem to have been satisfactorily presented before about the beginning of the nineteenth century and it seems to be entirely due to European mathematicians who, therefore, should have the credit of having completed the introduction of the negative numbers.

Among the decidedly vague historical statements relating to the negative numbers is the following "It was due to the influence of men like Vieta, Harriot, Fermat, Descartes and Hudde, however, that the negative number came to be fully recognized and understood." D. E. SMITH: *History of Mathematics*, Volume 2 (1925), page 259. The

undue credit given to R. Descartes (1596—1650) here and in many places as regards the development of the concept of negative numbers may be largely due to the fact that in such an eminently useful work of reference as the *Encyklopadie der Mathematischen Wissenschaften*, note 18 of the first article (1898), it is stated that the actual calculation with negative numbers begins with R. Descartes and that he attributed to the same letter, sometimes a positive and sometimes a negative value. Both of these statements are inaccurate ; but on account of the place in which they appear they will doubtless continue to exert a bad influence for many years unless their inaccuracy is given wide publicity in historical articles.

In the corresponding note (149) of the French edition of this encyclopaedia, which began to appear in 1904, it is stated, on the contrary, that the systematic calculations with negative numbers are posterior to R. Descartes, and on page 100 of J. Tropfke's *Geschichte der Elementar Mathematik*, Volume 2 (1933), it is also stated that R. Descartes did not attribute sometimes a positive and sometimes a negative number to the same letter unless he had placed a dot before this letter. In fact, no definite credit is due to any of the four men, Vieta, Harriot, Fermat or Descartes for the development of the concept of negative numbers.

One of the most striking evidences of the fact that the conception of negative numbers was not fully completed before about the close of the eighteenth century, is the view that every negative number is greater than infinity which was apparently endorsed even in the second half of the eighteenth century by L. Euler, who was one of the greatest mathematicians of all times and corresponded with leading mathematicians of his day. In fact, in the latter half of the eighteenth century, there was so much written in opposition to the use of negative numbers by various European mathematicians that one might say that there was then a real crusade in Europe against the use of these numbers. It is interesting to note in this connection that the real reason which inspired the work of the noted French mathematician, L N. M. Carnot (1753—1823), along the line of projective geometry was his aversion for negative numbers which he rejected because he thought that their use led to erroneous conclusions. This was done even in the early part of the nineteenth century. *Cf. Encyclopedie des Sciences Mathematiques*, Tome 3, Volume 2, page 3.

While the Hindus seem to have made the first correct use of negative numbers—and these numbers were used long before their theory was fully understood—the beginning of their history seems to antedate this correct use of them and to extend back at least as far as the operations with binomials such as $(a-b)^2$, where a exceeds b, by the ancient Greeks. The correctness of the fundamental rules that the product of an added number and a subtracted number is a subtracted number, and that the product of a subtracted number and a subtracted number is an added number, was established geometrically by the ancient Greeks for the cases when the minuends are greater than the subtrahends, as in the case of $(10-4)(8-3)$, and these rules were later seen to lead to correct results in many special cases where the minuends are less than the subtrahends and no case was observed where they lead to incorrect results. Hence they were gradually assumed by many to be universally true long before a proof of this fact appeared in the literature.

Hence F. Klein could almost correctly remark that "one must not think that the negative numbers are the invention of some clever man who manufactured them together with their consistency, perhaps out of the geometric representation. Rather, during a long period of development, the use of negative numbers forced itself, so to speak, upon mathematicians. Only in the nineteenth century, after men had been operating with them for centuries, was the consideration of their consistency taken up." *Elementary Mathematics from an advanced standpoint*, 1932, page 25. As a matter of fact, Kant considered negative numbers in an article as early as 1763 from practically the modern point of view and tried to make them useful in philosophy. *Cf. Zeitschrift fur mathematischen und naturwissenschaftlichen Unterricht*, 1924, page 112.

On page 23 of the volume noted in the preceding paragraph, F. Klein makes the following somewhat vague statement: "The creation of negative numbers is motivated, as you know, by the demand *that the operation of subtraction shall be possible in all cases*." One would naturally infer from this that the idea of making subtraction always possible was a prominent mathematical idea at the time, when negative numbers began to be introduced. As a matter of fact, this is not the case. The ideas of a mathematical group and of a mathematical

domain were among the latter ideas to become explicitly vitalizing mathematical concepts. If F. Klein had said that the demand that the operation of subtraction shall be possible in all cases was one of the later powerful motives for the use of negative numbers, while earlier powerful motives therefor were their usefulness in the theory of equations, in trigonometry and in analytic geometry, he would have given here a much clearer insight into the history of these numbers.

The creation of the negative numbers was motivated by many additional things. Among these is the exponential and logarithmic motive resulting from the comparison of geometric and arithmetic series. The desirability of using different numbers to express the numerical distances from a given starting point in opposite directions made itself felt early as may be seen from the early steps towards the development of analytic geometry. Instead of agreeing with F. Soddy that incredible confusions followed from the invention of negative numbers it seems much wiser to say that the introduction of these numbers saves us from the consideration of a large number of special cases and thus it greatly simplifies the intellectual penetration into various fields. It is true that the freedom from the consideration of special cases implies, like freedom elsewhere, responsibilities as regards caution. The fact that a problem has a numerical solution does not now impose the same conditions as it did when the number concept was less general. In fact, a mathematical solution is usually only a partial solution, but it is very suggestive. Mathematics is abstract while the world is concrete, and hence there is usually much left to be done after the mathematician has contributed his part towards the solution of a problem.

The three points to which we referred at the opening of this article can now be briefly answered as follows. The Hindus used negative numbers correctly at least as early as the times of Brahmagupta, who was born in 598; and they also possessed a correct theory of these numbers as regards addition and subtraction, but there is no definite evidence that a correct theory as regards the multiplication and division of these numbers was developed before about the beginning of the nineteenth century. The latter theory is a special case of the corresponding theory relating to complex numbers and it was developed simultaneously with this more general theory. The introduction of negative numbers as a permanent element of mathematics was

inaugurated by the use of these numbers by the Hindus and was later inspired by the many simplifications which were effected by this use in various fields of mathematical developments. This introduction met with considerable opposition from time to time until a satisfactory theory of the negative numbers was developed around the beginning of the nineteenth century and thus completed their introduction more than a thousand years after this introduction was inaugurated.

The introduction of the negative numbers belongs to the extensive field of mathematical developments which are due to a failure to state all the restrictions involved in certain early proofs. The history of these developments has never been written and involves many difficulties; but the general history of mathematics is much enriched by noting that many of the early proofs apply to wider fields than the authors of these proofs had in mind. When Diophantus first stated that the product of a subtracted number by a subtracted number is an added number, he naturally had no idea of the great generality of this rule, just as J. L. Lagrange had no idea of the generality of the so-called Lagrange theorem in group theory. *Many mathematical reputations are largely* due to the later growth of theorems and theories rather than to their observed generality at the time when they were first announced by those to whom they are now credited. Negative numbers entered mathematics as useful servants and later were accepted as staunch citizens after their real merits became known.

ON POWERS OF NUMBERS.

By K. Subba Rao, *Vizagapatam.*

Introduction. If there exists a non-trivial solution of

$$\sum_{s=1}^{m} x_s^k = \sum_{t=1}^{n} y_t^k \qquad \dots \quad (1)$$

where the x's and y's are positive integers, I write $(m) \overset{k}{=} (n)$.

If the equation (1) has an infinity of non-trivial posititive integral solutions, I write* $(m) \overset{k}{=} (n)$ i.o.

* i.o. ≡ infinitely often.

A non-trivial solution of (1) means one in which in particular

$$(x_1, \dots x_m, y_1, \dots y^n) = 1.$$

Further, I denote by $\beta = \beta(k)$, the least number β such that

$$\sum_{s=1}^{\alpha} x_s^k = \sum_{t=1}^{\beta} y_t^k \qquad (\alpha < \beta) \qquad \dots \quad (2)$$

has at least one non-trivial solution $(x_s, y_t > 0)$. Also

$$x_1, x_2, \dots \dots \stackrel{k}{=} y_1, y_2, \dots \dots$$

means that $\quad x_1^k + x_2^k + \dots \dots = y_1^k + y_2^k \dots \dots$

$r'_{s,k}(n)$ denotes the number of primitive representations of n as a sum of s positive kth powers.

In this paper, I prove

Theorem 1: $\beta(6) \leqslant 5$.

This is an improvement on the known result $\beta(6) \leqslant 16$ *.

Theorem 2: There exist integers n for which $r'_{5,6}(n) \geqslant 3$. †

Theorem 3:

$$(4) \stackrel{6}{=} (4)$$

$$(4) \stackrel{6}{=} (5)$$

$$(4) \stackrel{6}{=} (6)$$

$$(5) \stackrel{6}{=} (5)$$

$$(5) \stackrel{6}{=} (6)$$

The second of these equations obviously implies Theorem 1.

Theorem 4: $\quad (1) \stackrel{6}{=} (21)$, *i.o.*‡

---

* See DICKSON: *History of the Theory of Numbers*, Vol. 2, p. 683. Also S W. P. STEEN, *Proc. Lond. Math. Soc.* (Ser. 2), Vol. 29, 333-4. It seems not impossible that $\beta(k) \leqslant k-1$, $(k > 3)$.

† I have shown (in an as yet unpublished paper) that $r'_{5,6}(n) \geqslant 4$ has infinitely many non-trivial solutions.

‡ I have been unable to replace '21' by a smaller number, though it is likely that a much smaller number will do, perhaps '7'.

We have

$$23, 16, 11, 11 \overset{6}{=} 22, 19, 13, 12, 7$$

$$19, 19, 17, 9 \overset{6}{=} 21, 16, 15, 13, 1$$

which examples prove *Theorem 1*.

Theorem 2: There exist integers n for which $r'_{5,6}(n) \geqslant 3$, namely,*

$69, 45, 44, 38, 6 \overset{6}{=} 66, 57, 46, 20, 9 \overset{6}{=} 69, 46, 45, 30, 20$

$115, 75, 50, 23, 10 \overset{6}{=} 110, 95, 22, 19, 3 \overset{6}{=} 110, 95, 23, 15, 10$

$230, 150, 100, 69, 45 \overset{6}{=} 220, 190, 66, 57, 9 \overset{6}{=} 220, 190, 69, 45, 30$

$253, 165, 115, 75, 50, \overset{6}{=} 242, 209, 95, 33, 15 \overset{6}{=} 253, 165, 110, 95, 15$

$506, 437, 66, 57, 9 \overset{6}{=} 529, 345, 230, 45, 30 \overset{6}{=} 506, 437, 69, 45, 30$

$418, 361, 69, 45, 30 \overset{6}{=} 437, 285, 190, 66, 9 \overset{6}{=} 418, 361, 66, 57, 9$

$484, 418, 69, 45, 30 \overset{6}{=} 506, 330, 220, 579 \overset{6}{=} 484, 418, 66, 57, 9$

$506, 361, 330, 220, 57 \overset{6}{=} 484, 437, 285, 190, 66 \overset{6}{=} 484, 418, 361, 66, 57$

$529, 345, 330, 230, 220 \overset{6}{=} 484, 437, 418, 69, 66 \overset{6}{=} 506, 437, 330, 220, 69$

$529, 330, 285, 230, 45 \overset{6}{=} 506, 437, 225, 150, 69 \overset{6}{=} 529, 345, 230, 225, 150$

$529, 345, 285, 230, 190 \overset{6}{=} 506, 418, 361, 69, 57 \overset{6}{=} 506, 437, 285, 190, 69$

$418, 361, 330, 57, 45 \overset{6}{=} 437, 345, 225, 190, 150 \overset{6}{=} 437, 330, 285, 190, 45$

$506, 345, 225, 220, 150 \overset{6}{=} 484, 418\ 285, 66, 45, \overset{6}{=} 506, 330, 285, 220, 45$

$506, 437, 150, 100, 69 \overset{6}{=} 529, 345, 220, 190, 30 \overset{6}{=} 529, 345, 230, 150, 100$

---

* These examples have been derived from the example $22, 19, 3 \overset{6}{=} 23, 15, 10$. I should add that the existence of integers n for which $r'_{k,k}(n) \geqslant 2$ $(k > 6)$ has not been proved.

The following examples prove Theorem 3:—

$$20, 13, 13, 9 \overset{6}{=} 19, 17, 12, 5$$

$$19, 19, 17, 9 \overset{6}{=} 21, 16, 15, 13, 1$$

$$23, 16, 11, 11 \overset{6}{=} 22, 19, 13, 12, 7$$

$$19, 17, 10, 1 \overset{6}{=} 20, 14, 9, 7, 4, 3$$

$$17, 13, 13, 1 \overset{6}{=} 16, 16, 7, 7, 3, 3$$

$$19, 18, 17, 5 \overset{6}{=} 21, 15, 14, 8, 8, 3$$

$$21, 16, 14, 11, 11 \overset{6}{=} 22, 7, 7, 3, 2$$

$$20, 19, 12, 7, 6 \overset{6}{=} 21, 16, 15, 8, 2$$

$$18, 18, 13, 13, 5 \overset{6}{=} 20, 15, 11, 9, 2$$

$$18, 18, 11, 5, 5 \overset{6}{=} 19, 15, 15, 2, 2$$

$$20, 11, 11, 9, 5 \overset{6}{=} 19, 15, 13, 13, 2$$

$$20, 14, 11, 1, 1 \overset{6}{=} 19, 17, 10, 10, 7$$

$$16, 16, 10, 7, 3 \overset{6}{=} 17, 14, 12, 5, 4$$

$$21, 13, 13, 12, 11 \overset{6}{=} 20, 17, 15, 9, 7$$

$$19, 10, 10, 8, 6 \overset{6}{=} 18, 14, 13, 12, 2, 2$$

$$19, 16, 16, 6, 2 \overset{6}{=} 18, 18, 14, 13, 8, 4$$

$$19, 19, 13, 12, 11 \overset{6}{=} 20, 16, 15, 15, 7, 1$$

$$19, 16, 13, 3, 2 \overset{6}{=} 20, 12, 10, 9, 7, 5$$

$$19, 16, 16, 4, 2 \overset{6}{=} 20, 14, 13, 12, 10, 8$$

$$19, 19, 16, 6, 1 \overset{6}{=} 20, 18, 14, 13, 9, 5$$

$$16, 13, 11, 9, 2 \overset{6}{=} 15, 15, 10, 7, 4, 4$$

$$22, 7, 3, 3, 2 \overset{6}{=} 21, 15, 14, 13, 12, 10$$

$$20, 20, 13, 4, 3 \overset{6}{=} 22, 15, 14, 9, 2, 2$$

$$20, 20, 13, 9, 4 \overset{6}{=} 22, 15, 14, 10, 6, 5$$

$$20, 20, 10, 1, 1 \overset{6}{=} 22, 14, 13, 12, 8, 5$$

$$13, 13, 13, 12, 12 \overset{6}{=} 15, 14, 10, 9, 3, 2$$

$$23, 16, 15, 13, 2 \overset{6}{=} 22, 20, 12, 9, 7, 5$$

$$18, 17, 13, 10, 5 \overset{6}{=} 19, 16, 7, 6, 4, 3$$

$$19, 17, 9, 8, 2 \overset{6}{=} 20, 13, 12, 7, 6, 1$$

$$19, 16, 13, 6, 1 \overset{6}{=} 18, 17, 14, 12, 5, 5$$

$$22, 18, 17, 13, 5 \overset{6}{=} 23, 16, 15, 7, 6, 4$$

$$36, 13, 8, 8, 5 \overset{6}{=} 32, 32, 18, 9, 7, 4$$

$$40, 36, 19, 10, 6 \overset{6}{=} 42, 30, 20, 18, 13, 4$$

We have

$$(6t^6+1)^6 = (6t^6-1)^6+2(6t^5)^6+2(4t^3)^6+7(2t^3)^6+(2t)^6+8(t^6)$$

The right hand side contains *twenty-one* sixth powers and Theorem 4 is proved.

[Received 27-11-1933.]

A BIQUADRATE AS THE SUM OF FOUR BIQUADRATES.*

By S. SASTRY,

Lecturer in Mathematics, Benares Hindu University.

1. In confirmation† of Euler's conjecture that a fourth power can be the sum of four fourth powers R. Norrie found in 1910 by a series of special assumptions the single result ‡

$$353^4 = 30^4 + 120^4 + 272^4 + 315^4.$$

I shall give here a simple method of deriving this interesting result. It may be possible to use this method to derive other examples of a fourth power as the sum of four fourth powers.

---

* I wish to express my sincere thanks to Dr. S Chowla for reading the manuscript and for suggesting corrections.

† DICKSON: *History of the theory of Numbers*, Vol. 2, p. 652. Norrie's paper has been inaccessible to me.

‡ In a paper communicated to the London Mathematical Society, I gave the corresponding example for fifth powers, namely,

$$107^5 = 43^5 + 7^5 + 57^5 + 80^5 + 100^5.$$

Recently* Chowla and I have proved that a fifth power can be the sum of five positive fifth powers infinitely often. The corresponding result has not yet been proved for fourth powers.

2 We have

$$(a+b+c)^4-(a+b)^4=c(2a+2b+c)\{2a^2+2b^2+c^2+4ab+2ac+2bc\}.$$

Changing a to a^4, b to b^4 and c to c^4, we have

$$(a^4+b^4+c^4)^4-(a^4+b^4)^4$$

$$= c^4(2a^4+2b^4+c^4)\{2a^8+2b^8+c^8+4a^4b^4+2a^4c^4+2b^4c^4\} \qquad \ldots \quad (1)$$

My method is to express $2a^4+2b^4+c^4$ as a fourth power and $2a^8+2b^8+c^8+4a^4b^4+2a^4c^4+2b^4c^4$ as the sum of three fourth powers.

The first is always possible by means of E. Fauquemberge's identity†

$$(4x^4+y^4)^4=(4x^4-y^4)^4+2(4x^3y)^4+2(2xy^3)^4$$

if we take $a=4x^3y$, $b=2xy^3$ and $c=4x^4-y^4$. The second is also possible in the particular case $x=y=1$, *i.e.*, when $a=4$, $b=2$, and $c=3$. In this case,

$$2a^8+2b^8+c^8+4a^4b^4+2a^4c^4+2b^4c^4$$

$$=2(a^2)^4+2(b^2)^4+(c^2)^4+3(ab)^4+2(ac)^4+2(bc)^4-b^4+b^4+(ab)^4$$

$$=(2.16^4+2.4^4+9^4+3.8^4+2.12^4+2.6^4-2^4)+2^4+8^4$$

$$= 21^4 \qquad +2^4+8^4$$

In this case, (1) becomes $353^4-272^4=3^4.5^4(21^4+2^4+8^4)$,

that is $$35=272^4+315^4+30^4+120^4.$$

---

* To appear elsewhere.

† See DICKSON : *History of the theory of Numbers*, Vol. 2, p. 650.

A BRIEF SKETCH OF HINDU TRIGONOMETRY.

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In ancient India, Trigonometry was developed as an adjunct to Astronomy. Hence we find that in works on Astronomy, the author before proceeding to give the Grahasphutam (the process of finding the true places of planets from the mean) tells us how to find the sines of various angles. There is this difference, however, that the "sine" of the Hindus does not represent a ratio, as with us, but a linear measure ment. Their standard unit of measuring sines of angles, or what is the same thing, of measuring lengths, is the radius of the circle; and since the radius of any circle is equal to the length of an arc substending an angle of $57^\circ\ 17'\ 45'' = 3438$ minutes approximately (radian), the Hindus make their radius equal to $3438'$, and whenever they refer to त्रिज्या in their works this length of $3438'$ is meant by them.*

Now, in the circle with centre C and radius CA equal to $3438'$, the half-chord AM represents the Hindu Sine of the angle $A\hat{C}M$, so that to get the modern sine, we have to divide the former by $R = 3438'$; and similarly to get the modern trigonometric formula out of those of the Hindus, we have to substitute $R \sin A$, $R \cos A$ respectively for Sin A and Cos A. Thus the Hindu Sine of 90° is 3438. It will be seen that the word त्रिज्या in Sanskrit meaning 'Radius,' has derived its meaning from the fact that it is the ज्या or sine of three rasis (rasi $= 30^\circ$, since the Ecliptic is divided into twelve equal divisions called rasis).

Now, since the Hindus knew Pythogoras theorem (Playfair and other orientalists assert that Pythogoras got this theorems and some others also from the Hindus), they were familiar with most of the elementary trigonometrical relations derivable from right-angled triangles, such as $\text{Cos}\,A = \sqrt{R^2 - \text{Sin}^2 A}$; $\text{Sin}\,30^\circ = \text{Cos}\,60^\circ = R/2$; $\text{Sin}\,45 = R/\sqrt{2}$, etc., etc., (*vide* slokas 4, 6, under *Jyotpatti* Goladhyaya, *Sidhanta Sriomani*). Bhaskara also gives the Sines of 36° and 18°, in the slokas.

त्रिज्याकृतीषुघातात् त्रिज्याकृतिवर्गघातस्य, ।
मूलोनादष्टहृतान्मूलं षट्त्रिंशदंशज्या ॥ (7)

* In this paper, we shall write the Hindu Sine and Cosine with capital initials and the usual Sine and Cosine (which are ratios) with small letters s and c.

षट्त्रिंशदंशजीवा तत्कोटिज्याकृतेभूणाम्
त्रिज्याकृतीषुघातात् मूलं त्रिज्योनितं चतुर्भक्तम्
अष्टादशभागानाम् जीवा स्पष्टा भवत्येवं ॥ (9)

(Jyotpatti)

which means "Deduct the square-root of five times the fouth-power of the radius from five times the square of the radius, and divide the remainder by eight; the square-root of the quotient will be the Sine of 36° or the Sine of 54°" or in symbols

$$\text{Sin } 36^\circ = \sqrt{\frac{5R^2 - \sqrt{5R^4}}{8}}$$

"To get the Sine of 18°, deduct the radius from the square-root of the product of the radius and five, and divide the remainder by 4; the quotient will be found to be the exact Sine of 18°, or in symbols."

$$\text{Sin } 18^\circ = (\sqrt{5R} - R) / 4$$

The more general relations between the trigonometric functions are given by Bhaskara in the following words:

क्रमोत्क्रमज्याकृतियोगमूलाद्दलं तदर्धांशकशिञ्जिनी स्यात् ।
त्रिज्योत्क्रमज्यानिहतेर्दलस्य मूलं तदर्धांशकशिञ्जिनी वा ॥ (10)
त्रिज्याभुजज्याहतिहीनयुक्ते त्रिज्याकृती तद्दलयोः पदे स्तः ।
भुजोनयुक्त्रिभखण्डयोः ज्ये (12)

(Jyotpatti)

which means

$$\text{Sin } \frac{A}{2} = \frac{1}{2}\sqrt{\text{Sin}^2 A + \text{Versin}^2 A} = \sqrt{\frac{R}{2}\text{ versine A}}$$

$$\text{Sin}\left(45 + \frac{A}{2}\right) = \sqrt{R^2 + R \text{ Sin A}} \Big/ \sqrt{2};$$

$$\text{Sin}\left(45 - \frac{A}{2}\right) = \sqrt{R^2 - R \text{ Sin A}} \Big/ \sqrt{2}$$

which correspond, in modern notation to

$$R \sin\frac{A}{2} = \frac{1}{2}\sqrt{R^2 \sin^2 A + (R - R\cos A)^2} = \frac{R}{2}\sqrt{2 - 2\cos A},$$

and

$$R \sin\left(45 + \frac{A}{2}\right) = \sqrt{R^2 + R^2 \sin A} \Big/ \sqrt{2} = R\sqrt{1 + \sin A} \Big/ \sqrt{2}.$$

Again when A and B are two arcs, Bhaskara gives the sine of their difference

$$\text{Sin}\,\tfrac{1}{2}(A - B) = \tfrac{1}{2}\left\{(\text{Sin } A - \text{Sin } B)^2 + (\text{Cos } A - \text{Cos } B)^2\right\}^{\frac{1}{2}}$$

in the following sloka :

' यद्दोर्ज्ययोरंतरमिष्टयोर्यत् कोटिज्ययोः तत् कृतियोगमूलं,
दलीकृतं स्यात् भुजयोर्वियोगखंडस्य जीवैवमनेकधावा " (13)

(*Jyotpatti*)

Bhaskara next gives the calculation of sines without having recourse to the extraction of roots in the following sloka.

दोर्ज्याकृतिर्व्यासदलार्थभक्ता लब्धत्रिमौर्व्योः
विवरेण तुल्या दोःकोटिभागांतरशिञ्जिनी स्यात्

which is equivalent to the formula $\text{Sin}(2A - 90) = (R^2 - 2\,\text{Sin}^2 A)/R$ ' In this way several sines may be found" he says and then proceeds. 'to give the rules for finding the sines of every degree from $1°$ to $90°$ In doing so, he utilizes the formulae

$$R \sin (A \pm B) = \text{Sin } A \cos B \pm \text{Cos } A \sin B.$$

which are given in the following verses

" चापयोरिष्टयोः दोर्ज्ये मिथः कोटिज्यकाहते,
त्रिज्याभक्ते तयोरैक्यं स्यात् चापैकस्य दोर्ज्यका,
चापान्तरस्य जीवा स्यात् तयोरन्तरसंमिता,
अन्यज्यासाधने सम्यगियं ज्याभावनोदिता."

The process of finding the sines of different angles, given above by Bhaskara, is termed by him "प्रतिभागज्यकाविधिः" which means "the

method of finding the sines of every degree," and can be summed up as follows. Suppose we are to find the Sine of 35°. We start with the angle which is nearest to the given angle whose sine is known; for instance 30°; then we use the formula given by the sloka

" स्वर्गांगेषु षडंशोनवर्जिता भुजशिञ्जिनी,
कोटिज्या दशभिः क्षुण्णा त्रिसप्तेषुविभाजिता,
तदैक्यमग्रजीवा स्यात् अंतरंपूर्वशिञ्जिनी."

" Subtract 1/6569th part of Sine 30° from the same: multiply the Cosine of 30° by 10 and divide by 573; add or subtract the resulting values; you will get $\text{Sin}\,(30 + 1^\circ)$ or $\sin\,(30 - 1^\circ)$." We proceed thus, and get the sine of 35°. This formula is equivalent to

$$R \text{ Sin } 31^\circ = \text{Sin } 30^\circ \text{ Cos } 1^\circ + \text{Cos } 30^\circ \text{ Sin } 1^\circ.$$

Here the value of Sine 1°, namely $60'$, is utilized and calculations are simplified taking advantage of the constant factors Sin 1° and Cos 1°.

The method given in *Surya Siddanta* for obtaining the sines of different angles is different from this. There the quadrant is divided into 24 equal parts, each containing $3^\circ 45'$ and the Sine of each is calculated starting with the initial Sine, namely that of $3^\circ 45'$ which is taken to be $225'$. This may be accounted for in two ways. It may be that the Hindu Siddhantis perceived that the sine of a very small angle would be equal to its circular measure; or, they might have actually calculated Sin $3^\circ 45'$ starting from Sin 30°. Judging from the fact that they knew the formula $\text{Sin}\,\frac{1}{2}A = \sqrt{\frac{1}{2} R \text{ versin } A}$, it is more probable that the ancients actually calculated the Sin of $3^\circ 45'$ and finding that it is very nearly equal to 225, adopted this as the initial sine. The formula adopted, is.

$$\text{Sin}\,(n+1)\,A = \text{Sin}\,nA + \text{Sin}\,A - \Big(\sum_{r=1}^{r=n} \text{Sin}\,rA\Big) / \text{Sin}\,A \ldots \quad \text{(i)}$$

contained in the following slokas of *Surya Siddhanta*, Chapter 2.

" राशिलिप्ताष्टमोभागः प्रथमं ज्यार्थमुच्यते,
तत्तद्विभक्तलब्धोनमिश्रितं तत् द्वितीयकं,
आद्येनैवं क्रमात्पिंडान् त्त्वालब्धोनसंयुताः
खण्डकास्युश्चतुर्विंमूश ज्यार्थपिण्डाः क्रमादमी."

The formula may be obtained thus

$$\text{Sin}\,(n+1)\,A + \text{Sin}\,(n-1)\,A = \text{Sin}\,nA\,(\text{Sin}\,2A\,/\,\text{Sin}\,A).$$

Replacing n by $n-1$, $n-2$, ... 1 in succession and adding all the results we have

$$\text{Sin}\,(n+1)\,A + 2\sum_{r=1}^{n}\text{Sin}\,rA - \text{Sin}\,nA - \text{Sin}\,A$$

$$= (\sum_{r=1}^{n}\text{Sin}\,rA)\,\text{Sin}\,2A\,/\,\text{Sin}\,A.$$

$$\text{Sin}\,(n+1)\,A = \text{Sin}\,nA + \text{Sin}\,A - (\sum_{1}^{r}\text{Sin}\,rA)(2\text{Sin}\,A - \text{Sin}^2A)\,/\,\text{Sin}\,A.$$

Now, finding that Sin A = 225′ and Sin 2A = 449′ (A = 3°45′) which values may have been obtained from Sin 30° as remarked above, they saw that the factor 2 Sin A — Sin 2A reduced to un ty so that the formula now takes the form given above.

Indeed, we can calculate the sines of consecutive angles from 1° upto the 90° utilizing the same formula in a different way. Instead of assuming Sin 30°45′ = 225′, we will be nearer the truth by assuming Sin 1° = 60′. Hence calculating Cos 1° from the formula $\text{Cos}\,\theta = \sqrt{R^2 - \sin^2\theta}\,/\,R$, we find Cos 1° = 3437·47′.

$\therefore$ Sin 2° = (2 Sin 1° Cos 1°)/3438 = 2 × 60 × 3437·47 / 343·8 = 119·9815′

Hence 2 Sin 1° — Sin 2° = ·0185′. Hence in the formula making A = 1°, and the factor 2 Sin 1° Sin 2° = ·0185′, the successive sines of 1°, 2°, 3°, etc., can be calculated easily Thus

Sin 1° = 60′	Sin 6° = 359·353′	Sin 11° = 655·765′
Sin 2° = 119·9815′	Sin 7° = 418·965′	Sin 12° = 714·562′
Sin 3° = 179·926′	Sin 8° = 478·447′	Sin 13° = 773·137′
Sin 4° = 239·815′	Sin 9° = 537·781′	Sin 14° = 831·490′
Sin 5° = 299·630′	Sin 10° = 596·782′	Sin 15° = 889·70′

The approximation will be found correct to five decimal places.

Spherical Trigonometry.

In spherical trigonometry, the Hindu Siddhantis were conversant with Napier's rules concerning right-angled spherical triangles and with various other formulae such as

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$
$$\sin c / \sin C = \sin b / \sin B = \sin a / \sin A, \text{ etc.}$$

Their methods may be best illustrated by a few examples, which have a bearing on Astronomy.

EXAMPLE 1. To find the Sun's altitude when the azimuth is 45°, given only one observation, namely, that of the rising amplitude* and *palabha*.† The rule given in the *Surya Siddhanta* reads :

त्रिज्यावर्गार्धतोग्रज्यावर्गोनाद् द्वादशाहतात् ।
पुनर्द्वादशनिघ्नाच्च लभ्यते यत्फलं बुधैः ॥
शङ्कुवर्गार्धसंयुक्तविषुवद्वर्गभाजितात् ।
तदेवकरणीनाम तांपृथक् स्थापयेत् बुधः ॥
अर्कघ्नी विषुवच्छायाग्रज्यया गुणिता तथा ।
भक्ता फलाख्यं तद्वर्गसंयुक्त करणीपदं ॥
फलेन हीनसंयुक्तं दक्षिणोत्तरगोलयोः ।
तत्रिज्यावर्गविश्लेषान्मूलं दृग्ज्याभिधीयते ॥

Surya Siddhanta, Chapter 3, *Slokas*, 28, 29, 30, 32.

" Sin (Sun's altitude) $= (\text{Karani} + \text{phala}^2)^{\frac{1}{2}} \mp \text{phala}$,

where

$$\text{Karani} = 12^2 \left(\tfrac{1}{2} R^2 - \text{Sin}^2 A\right) / \left(\tfrac{1}{2} 12^2 + \text{palabha}^2\right), \text{ and}$$
$$\text{phala} = (12 \text{ palabha} \times \text{Sin } A) / \left(\tfrac{1}{2} 12^2 + \text{palabha}^2.\right)$$

where $(A = 90 - \text{azimuth})$."

To prove the truth of this formula, let us take the spherical triangle PZS, in which P is the pole, Z the zenith and S the sun. If

$PS = 90 - \delta$; $PZ = 90 - \phi$: $ZS = z$; $\angle PZS = 360° - \alpha$;
we have $\sin \delta = \cos z \sin \phi + \sin z \cos \phi \cos \alpha$

* Amplitude is used by the Hindus in place of 90°—azimuth.

† *Palabha* is the shadow cast by a gnomon of 12 units height at noon on a day of an equinox.

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where α is measured from the north point, so that the amplitude is here $180 - \alpha$. Transforming this formula into the Hindu one, it becomes

$$R^2 \operatorname{Sin} \delta = R \operatorname{Cos} z \operatorname{Sin} \phi + \operatorname{Sin} z \operatorname{Cos} \phi \operatorname{Cos} \alpha.$$

Substituting in this

$$\sin \phi = (\text{palabha} \times R) \,/\, \text{hypotenuse}^* = pR/H$$

$$\operatorname{Cos} \phi = (R \times \text{gnomon}) \,/\, \text{hypotenuse} = Rg/H$$

and $\alpha = 180 - 45^\circ = 135^\circ$,

we have $$\operatorname{Sin} \delta = \frac{p \operatorname{Cos} z}{H} - \frac{g \operatorname{Sin} z}{H\sqrt{2}}.$$

But, from the spherical triangle SnP (see figure next page) where S is the rising Sun, n the north point of the horizon, $Sn = 90 - A$ where A is his rising amplitude, and $SP = 90 - \delta$, $Pn = \phi$,

$$\operatorname{Sin} \delta = \operatorname{Cos} \phi \operatorname{Sin} A/R = g \operatorname{Sin} A/H.$$

Substituting this value of $\operatorname{Sin} \delta$ in the above formula

$$\frac{g \operatorname{Sin} A}{H} = \frac{p}{H} \operatorname{Cos} z - \frac{g \operatorname{Sin} z}{H\sqrt{2}}.$$

But since $\operatorname{Sin}^2 z = R^2 - \operatorname{Cos}^2 z$, this reduces to

$$p \operatorname{Cos} z - g \sqrt{R^2 - \operatorname{Cos}^2 z} \,/\, \sqrt{2} = g^2 \operatorname{Sin} A$$

i.e., $$(p^2 + \tfrac{1}{2} g^2) \operatorname{Cos}^2 z - 2 \operatorname{Cos} z \, (pg \operatorname{Sin} A) = g^2 (\tfrac{1}{2} R^2 - \operatorname{Sin}^2 A)$$

i.e., $$\operatorname{Cos}^2 z - 2 \operatorname{Cos} z \times \text{phala} = \text{Karani}$$

by substituting phala and Karani respectively for

$$\frac{12\, p \operatorname{Sin} A}{72 + p^2} \quad \text{and} \quad \frac{g^2 (\tfrac{1}{2} R^2 - \operatorname{Sin}^2 A)}{72 + p^2}; \qquad \text{since } g = 12$$

i.e., $\operatorname{Cos} z = \sqrt{\text{Karani} + \text{Phala}^2} \mp \text{phala}$ as is given in the *Surya Siddhanta*.

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* The fundamental triangle used by the Hindus is the right-angled triangle formed of the gnomon, its shadow and the hypotenuse. It is called the प्रधानाक्षक्षेत्र Thus, if AB represents the gnomon, BC the equinoctial shadow, and AC the hypotenuse, $\angle A = \phi$ represents the latitude; hence $\sin \phi/R$ = palabha / hypot.

EXAMPLE 2. To find the Sun's altitude when h (hour angle), δ and ϕ are given. The rule given in the *Surya-Siddhanta* is

" त्रिज्योदक् चरजायुक्ता याम्यायां तद्विवर्जिता, अंत्या नतोत्क्रमज्योना स्वाहो-रात्रार्धसंगुणा. त्रिज्याभक्तो भवेत् छेदो लम्बज्याघ्नोऽथभाजितः त्रिभज्यया भवेत् शंकुःतद्वर्गं परिशोधयेत् त्रिज्यावर्गात् पदं दृग्ज्या, ...

i.e., $$R \pm \text{Sin (chara)} = \text{Antya};$$

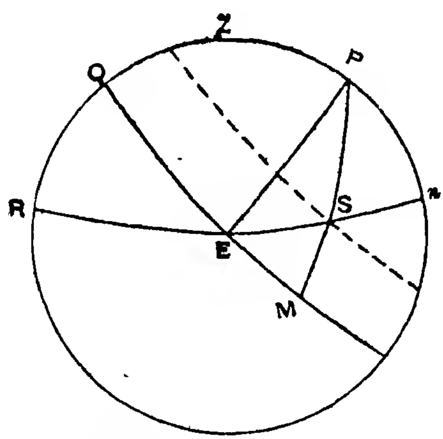
$$(\text{Antya} - \text{Vers}\, h)\ \text{Cos}\, \delta / R = \text{cheda};$$

$$(\text{cheda} \times \text{Cos}\, \phi) / R = \text{Sanku} = \text{Sin (Sun's altitude)}.$$

Hence

$$\text{Sin (Sun's altitude)} = \{ R \pm \text{Sin (chara)} - \text{Vers}\, h \}\ \text{Cos}\, d\, \text{Cos}\, \phi / R^2$$

The arc EM in the figure, which is the portion of the equator intercepted between the declination circle of the Sun while rising and the fixed circle PE, is called *chara* or Ascential difference. The fixed



circle PE is called उन्मण्डल or the rising circle. From the right-angled spherical triangle EMS in which MS is the declination of the Sun, EM the Ascential difference and the angle SEM $= 90 - \phi$, we have by Napier's rules $\sin(\text{chara}) = \pm \tan \delta \tan \phi$, or with the Hindu

notation

$$\text{Sin (chara)} = \pm R \tan \delta \tan \phi. \qquad (1)$$

Also from the triangle PZS

$$\cos z = \sin \delta \sin \phi + \cos \delta \cos \phi \cos h$$

which in Hindu Trigonometry becomes

$$\text{Cos } h = R^2 \text{ Cos } z / \cos \phi \text{ Cos } \delta \mp R \tan \phi \tan \delta. \qquad (2)$$

Hence by addition from equations (1) and (2)

$$\text{Cos } h \pm \text{Sin (chara)} = R^2 \text{ Cos } z / \text{Cos } \phi \text{ Cos } \delta.$$

$$\therefore \qquad \{ \text{Cos } h \pm \text{Sin (chara)} \} \text{ Cos } \phi \text{ Cos } \delta / R^2 = \text{Cos } z$$

which is also Sin (Sun's altitude). This is the formula given in the *Surya Siddhanta.*

EXAMPLE 3. To find the true *Valana*, (*i.e.*, the angle which determines the points of the disc of the eclipsed body where the eclipse commences and where it ends).

In *Surya Siddhanta* the first Sloka in Chapter 9, remarks

नच्छेद्यकमृते यस्मात् भेदा ग्रहणयोः स्फुटाः ।
ज्ञायंते तत् प्रवक्ष्यामि छेद्यकज्ञानमुत्तमं ॥

i.e., " the phases of the Eclipses cannot be understood without knowing the method of projection (छेद्यकविधि)."

The angle called *Sphuta Valana* is the angle between Mn and MK where M is the centre of the eclipsed body, the Sun or Moon, P the celestial pole, and K the Kadamba or the pole of the Ecliptic and n the north point of the horizon. This angle is equal to the sum or difference of the angles PMn and KMP. The $\angle$PMn is called the *Aksha Valana* and the $\angle$KMP is called the *Ayana Valana.*

From the triangle PMn,

$$\sin (\text{Aksha Valana}) / \sin \phi = \sin \text{P}n\text{M} / \cos \delta$$

and from the triangle KMP,

$$\sin(\text{Ayana Valana}) / \sin \omega = \cos \lambda / \cos \delta$$

(where λ is the longitude of the centre of the eclipsed body M; and ω = obliquity of the ecliptic = 24° according to the Hindu works).

The *Surya Siddhanta* gives the same formulae, except that it has the radius of the diurnal circle in the place of $\text{Cos}\,\delta$. But both are identical because, if $E'Q'$ represent an arc of the diurnal circle parallel to the Equator, $E'Q' / EQ = \text{Cos}\,\delta / R$.

$$\text{But} \quad \frac{E'Q'}{EQ} = \frac{\text{diurnal circle}}{\text{Equator}} = \frac{2\pi \times \text{radius of the diurnal circle}}{2\pi R}.$$

Hence $\text{Cos}\,\delta$ is equal to the radius of the diurnal circle which is called द्युज्या in Sanskrit. The sum of the angles Aksha Valana and Ayana Valana determines the angle which gives the means of projecting the line of the Ecliptic upon the disc of the Eclipsed body, the Sun or the Moon.

GLEANINGS

19. The boy or girl should be accustomed to use language like a rapier, should despise a clumsiness in inference as if it were a foul in a game. The classes should collect blunders, disingenuous statements and false conclusions from public discussions. They should botanise for errors and bring the precious finds to the class room for dissection. They will also go as far as their willingness and aptitudes will take them in the exact and rigid reasoning processes of Mathematics

H. G. WELLS : *The Work, Wealth and Happiness of Mankind*, page 759.

20. What is physical is subject to the laws of mathematics and what is spiritual to the laws of God, and the laws of Mathematics are but the expression of the thoughts of God.

THOMAS MILL : The uses of Mathesis; Bibliotheca Sacra;
Extracted from *Memorabilia Mathematica*, No 275.

21. A story is told by Einstein against himself that comes in amusingly here. He is a great mathematician but he is not a ready reckoner, and during the crisis of the mark he thought that a tram conductor had given him back too much change—a hundred thousand marks or so—and had to be convinced of his error. "Everybody," said the kindly tram conductor, "hasn't the gift of calculation with these big figures. I must not take advantage of you .. " And Laplace, one of the greatest mathematicians the world has ever known, was dismissed from the Ministry of the Interior by Napoleon for the grossest incompetence. But our organising world requires these exceptional individuals far more than it does an endless multitude of fairly good-all-round people.

H. G. WELLS : *The Work, Wealth and Happiness of Mankind*, page 680.

NOTES AND DISCUSSIONS.

The Editor welcomes for publication under this heading, brief discussions of interesting problems, critical comments, and suggestions likely to be helpful in the class-room.

Congruence Properties of Partitions.

Let $p(n)$ denote the number of unrestricted partitions of n. Ramanujan* conjectured that

If $\delta = 5^a 7^b 11^c$ *and* $24\lambda \equiv 1 \pmod{\delta}$, *then*

$$p(\lambda),\ p(\lambda + \delta),\ \ldots\ \ldots \equiv 0 \pmod{\delta}.$$

A recent table † by H. Gupta gives

$$p(243) = 133\ 978\ 259\ 344\ 888.$$

Hence

Ramanujan's conjecture is false for $\lambda = 243$, $\delta = 343$.

S. Chowla.

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On a certain Arithmetical Function.

Let $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ where the p's are distinct primes,

$$\lambda(1) = 1, \qquad \lambda(n) = (-1)^{\alpha_1 + \ldots + \alpha_r} \quad (n \geqslant 2),$$

$$L(x) = \sum_{n \leqslant x} \lambda(n).$$

Po'lya‡ has verified that for $2 \leqslant x \leqslant 1500$,

$$L(x) \leqslant 0. \qquad \ldots \quad (1)$$

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* *Proc. Camb. Phil. Soc*, XIX (1919), 207—210.

† This table gives $p(n)$ for $n \leqslant 300$. With the help of Macmohan's table (extending to $n \leqslant 200$) I have verified Gupta's table upto $n = 243$.

‡ *Jahresberichte der Deutsche Math. Vereinigung*, 28, 1919, 38—40.

I have tabulated* the values of $L(x)$ for all $x \leqslant 10000$ *and find that* (1) *is true up to this limit.*

The values of $L(x)$ for $x = 1000, 2000, \ldots, 10000$ are $-14, -22, -44, -52, -46, -56, -100, -44, -46, -96$.

Further, the maximum of $L(x)$ for $1{,}500 \leqslant x \leqslant 10{,}000$ occurs at $x = 3281$ when $L(x) = -7$, while $L(x)$ is, in the same range, minimum for $x = 9840$ when $L(x) = -130$.

S RAJARAMAN.

[Received 24-2-34].

Notes on the Theory of numbers (II). Remarks on the preceding Paper.

The following are some of the outstanding 'asymptotic' properties of $L(x)$.

(I) $$L(x) \leqslant 0 \qquad (x \geqslant 2)$$

implies† *the Riemann hypothesis on the roots of the zeta-function*

(II) $$L(x) - \frac{\sqrt{x}}{\zeta(\frac{1}{2})} = \Omega_{\substack{R\\L}}(x^{\frac{1}{2}-\delta}), \quad \textit{for every } \delta > 0$$

[Note that $\zeta(\frac{1}{2}) < 0$].

(III) $$L(x) \sim \sqrt{x} / \zeta(\tfrac{1}{2}) \quad \textit{is false}‡.$$

(I) and (II) follow from Landau, *Handbuch*, Satz (S. 698).

(III) follows from Landau's *Vorlesungen*, II, S. 162.

S. CHOWLA.

[Received 24-2-34].

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* At the suggestion of Dr. S. Chowla.

† This was pointed out by Dr. A. Weil, who drew my attention to the problem (1) is mentioned by Polya, *loc. cit.*

‡ A proof of (III) was indicated to me by Dr. Walfisz.

On Euler's discussion of the equation

$A^4 + B^4 + C^4 + D^4 = E^4$

In this note, I show that Euler's discussion* of the problem

to find four biquadrates whose sum is a biquadrate

is apparently pointless.

Evidently $A^4 + B^4 + C^4 + D^4 = E^4$

if $A^2 = (p^2 + q^2 + r^2 - s^2)/n$, $B^2 = 2ps/n$,
$C^2 = 2qs/n$, $D^2 = 2rs/n$, $E^2 = (p^2 + q^2 + r^2 + s^2)/n$.

These five functions are to be made squares. This will be true of the first and last if

$$(p^2 + q^2 + r^2)/n = a^2 + b^2, \quad s^2/n = 2ab \quad \dots \quad (1)$$

Then $s^2 = 2abn$ = a square, if $2n = lm$, $a = lf^2$, $b = mg^2$, whence $a = lmfg$. Next,

$$2ps/n = 4pfg = 4x^2, \quad 2qs/n = 4qfg = 4y^2, \quad 2rs/n = 4rfg = 4z^2,$$

f $\qquad p = x^2/fg, \quad q = y^2/fg, \quad r = z^2/fg.$

Substituting these values into (1) we get

$$x^4 + y^4 + z^4 = \tfrac{1}{8}ab(a^2 + b^2) \dagger \quad \dots \quad \dots \quad \dots \quad (2)$$

This is the final condition arrived at by Euler, and no discussion of this condition is given.

The right hand side of (2) is exactly equal to $\left(\frac{a+b}{2}\right)^4 - \left(\frac{a-b}{2}\right)^4$ and therefore the condition (2) can be written

$$x^4 + y^4 + z^4 + \left(\frac{a-b}{2}\right)^4 = \left(\frac{a+b}{2}\right)^4$$

which is of the same type as the original form, and hence Euler's discussion leads nowhere.

Vizagapatam K. SUBBA RAO.

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* Dickson's *History of the Theory of Numbers*, Vol. II, p. 649.

† The above discussion is from Dickson, *loc. cit.*

On some Loci connected with a Triangle.

1. Given the base BC and the vertical angle A of a triangle, the following loci are discussed in text-books:

(1) *The Ortho-centre.* It is frequently stated that the locus is the arc of a segment of a circle on BC containing an angle equal to the supplement of A. The statement is erroneous, being incomplete when A is acute, and only partially true when A is obtuse. The actual locus is the arc of a segment of a circle containing an angle equal to A, described on a line equal and parallel to the given base and at a determinable distance from it.

(2) *The Ex-centre* I_1. The angle $BI_1C = 90° - A/2$ and therefore I_1 is on the segment of a circle on BC subtending an angle $90° - A/2$. The whole arc of this segment, however, does not form the locus. As I_1 lies on the external bisectors of the angles ABC and ACB, BI_1 and CI_1 cannot make with BC angles greater than a right angle. Hence I_1 cannot lie outside the region formed by the perpendiculars through B and C to BC. This restriction does not appear to have been noted before.

(3) *The Ex-centres* I_2 *and* I_3. I_2 and I_3 lie on the segment of a circle on BC containing an angle $A/2$. In this case, the segment within the region of the perpendiculars through B and C to BC cannot form part of the locus. The locus consists of two equal (detached) segments of a circle, one of which represents the locus of I_2 and the other of I_3. For a fuller discussion of this locus reference may be made to *The Mathematical Gazette*, Vol. VIII, Notes, pp. 263—265.

Of course it may be said from considerations of symmetry, that the locus includes a symmetrical counterpart on the other side of BC in all the above cases.

2. THEOREM. The locus of a point which is such that the rectangle under its differences from the equal sides of an isosceles triangle is equal to the square on its distance from the base is the circle touching the equal sides at the extremities of the base.

The theorem will be true only if the equality is true both in magnitude and in sign. If, however, the magnitude alone is taken into consideration, the proposition and the usual demonstration are

incomplete, for the ex-centres of the triangle corresponding to the equal sides satisfy the given condition and must be on the locus. In this case the locus of the point using the trilinears is given by

$$(\alpha^2 - \beta\gamma)(\alpha^2 + \beta\gamma) = 0$$

giving the circle and a hyperbola, having double contact with it at the ends of the base.

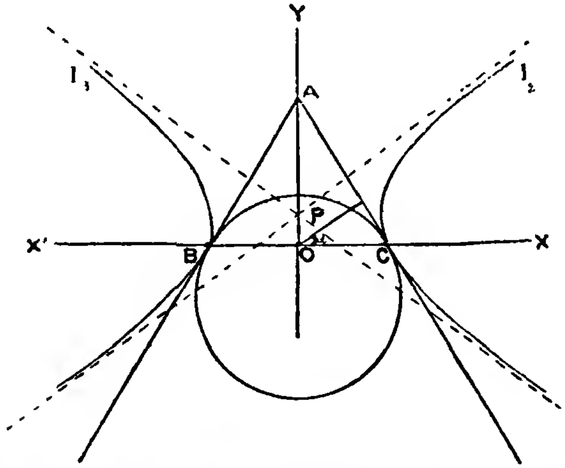
With the base and the bisector of the vertical angle of the triangle as co-ordinate axes, the equations to the sides may be written in the form $\pm x \cos\alpha + y \sin\alpha = p$. The equation to the required locus is then

$$y^2 = \pm (x \cos\alpha + y \sin\alpha - p)(y \sin\alpha - x \cos\alpha - p),$$

i.e., $x^2 + y^2 + 2py \tan\alpha \sec\alpha - p^2 \sec^2\alpha = 0$ a circle

and $x^2 \cos^2\alpha - y^2(1 + \sin^2\alpha) + 2py \sin\alpha - p^2 = 0$ a hyperbola

symmetrical about the y-axis. That the hyperbola passes through the ex-centres corresponding to the equal sides can be easily verified. The position of the centre and asymptotes and the magnitude of the axes and the eccentricity can be determined; the hyperbola is as drawn in the figure.



It may be remarked that as α varies from 90° to 0° the shape of the hyperbola changes, the eccentricity $\sqrt{2/(1 + \sin^2\alpha)}$ increasing from 1 to $\sqrt{2}$. In fact, given any hyperbola whose eccentricity is less

than $\sqrt{2}$, *i.e.*, whose transverse axis is greater than the conjugate axis, it is possible to find an isosceles triangle for which the hyperbola will be part of the locus considered.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad (a > b),$$

is the hyperbola and $y = \pm mx + c$ and $y = k$ the sides of the triangle. Comparing the equation of the hyperbola with

$$(y - mx - c)(y + mx - c) = -(1 + m^2)(y - k)^2$$

the values of m, k and c are

$$\pm \frac{b\sqrt{2}}{\sqrt{a^2 - b^2}}, \quad \pm b\sqrt{\frac{a^2 - b^2}{a^2 + b^2}} \quad \text{and} \quad \pm b\sqrt{\frac{a^2 + b^2}{a^2 - b^2}}.$$

It can be easily verified that the sides are tangents to the hyperbola at the points where the base meets it. Though these values apparently give two triangles, there is really only one triangle as they are symmetrically placed with respect to the x-axis.

If $a > b$, such a triangle is impossible. Hence all hyperbolas whose eccentricities lie between 1 and $\sqrt{2}$ may be considered as loci of points satisfying the condition $-p^2_t = p_b \cdot p_c$ for a properly chosen isosceles triangle. This result, I believe, is new.

Annamalai University,
Annamalainagar.

G. V. KRISHNASWAMI.

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22. If we compare a mathematical problem with an immense rock, whose interior we wish to penetrate, then the work of the Greek mathematicians appears to us like that of a robust stone-cutter who with indefatigable perseverance attempts to demolish the rock gradually from the outside by means of hammer and chisel; but the modern mathematician resembles an expert miner, who first constructs a few passages through the rock and then explodes it with a single blast, bringing to light its inner treasures.

HANKEL: Die Entwickelung der Mathematik in den letzen Jahrhunderten, (from *Memorabilia Mathematica*, (718).

On a property relating to an Ortho-centric tetrad.

1. It is known* that:

If from any point on a conic parallels be drawn to the diameters bisecting the sides of any inscribed triangle, the lines so drawn meet the corresponding sides of the triangle in three collinear points. ... (1.1)

In particular, when the four points form an ortho-centric tetrad on the conic, the conic is necessarily a rectangular hyperbola whose centre lies on the nine-point circle; and we have the result contained in question 1576,† *viz.*,

If O be any point on the nine-point circle of a triangle ABC whose ortho-centre is D and L, M, N the mid. points of BC, CA, AB, the parallels through D to OL, OM, ON meet the sides of ABC in three collinear points. ... (1.2)

It follows from (1.2) that if DX, DY, DZ is one triad of lines through D which meet the sides of ABC in three collinear points, then every triad of lines through D obtained by rotating the lines of the first triad through the same angle has this property.

Since the nine-point circle of ABC is symmetrically related to the ortho-centric tetrad ABCD, the property (1.2) relating to the triangle ABC is true also for the three other triangles formed out of ABCD. Thus we have four lines of collinearity L_1, L_2, L_3, L_4 by taking the triangles BCD, etc., respectively.

2. Let H be a unique rectangular hyperbola through ABCD having O for centre (the whole figure lying in a plane P). By projecting the points at infinity on H into the circular points at infinity on to the plane of projection p and denoting the projections by corresponding small letters, it is easily seen that a, b, c, d are points on a circle h in the plane p.

* Taylor: *Geometry of Conics*, p. 366.

† Vide *The Mathematics Student*, Vol. I, p. 76.

The lines L_1, L_2, L_3, L_4 project into the pedal lines l_1, l_2, l_3, l_4 of a, b, c, d respectively with respect to bcd, etc. Since these four pedal lines meet at a point t, which bisects the join of any one of the points a, b, c, d to the ortho-centres of the triangles formed by the other three, we have the corresponding result in plane P, *viz.*:

If ABCD is any ortho-centric tetrad whose common nine-point circle is Γ, the lines L_1, L_2, L_3, L_4 are concurrent at a point T. ... (2.1)

Taking the triangle ABC, we note that the circum-circle Ω of ABC in P projects into the unique rectangular hyperbola ω in p through a, b, c, i, j where i, j are the projections on p of the circular points at infinity in P. This rectangular hyperbola ω necessarily passes through the ortho-centre u of abc. Hence, it follows that in the plane P, the parallels through A, B, C to OL, OM, ON meet at a point U on Ω. Since t is the mid. point of du in p, T is the mid. point of DU in P. Since the join of any point on the circum-circle of a triangle to its ortho-centre is bisected by its nine-point circle, we see that T is a point on Γ. Also from the similar triangles ABC, LMN (with centre of similitude G) it follows that if S is the centre of Ω, OS is parallel to DU and equal to half DU, so that O and T are collinear with the nine-point centre of ABC. Hence we arrive at the conclusion that

as O moves on Γ, the corresponding point T also moves on Γ, so that OT is a diameter of Γ. ... (2.2)

If we take the point T as our starting point, the above process leads to O, as the point of concurrence, so that the relation between O and T is symmetric.

Annamalainagar. K. RANGASWAMI.

REVIEWS.

The Calculus of Finite Differences, by L. M. MILNE-THOMSON, M.A., F.R.S.E., Assistant Professor of Mathematics in the Royal Naval College, Woolwich. pp. xxiii+558. London. 1933. (Macmillan & Co.) *Price* 14*s*.

The necessity for a good text-book and an introduction to the neglected subject of Finite Differences has been felt in this country as in all others where English is the medium of mathematical instruction. With us, the long-vanished treatise of Boole still figures in the average syllabus, and the fact of its being out of print is hardly felt, as no one ever gives the calculus the attention that it deserves.

The author has kept both the above needs in mind, and so far as a text-book to supplant Boole is concerned, has succeeded admirably. The reason for this is seen in the first seven chapters, of which five are devoted to interpolation. Prof. Milne-Thomson's tables of elliptic functions fully prepare us for the excellence of this part. Chapters VI and VII treat Bernoullian and Eulerian polynomials, and finite differentiation; the writer himself considers these seven chapters a fair introduction to the subject. Had the work ended here, the student would have had a moderately priced hand-book, particularly useful to the computer. It is gratifying, for example, to see the Gaussian interpolation formula properly treated, whereas, in the usual order of things, it is fired off at the class as a property of Legendre polynomials.

Unfortunately, the rest of the work is "room ... for the more modern developments of the finite calculus." The treatise of Norlund might be too heavy for the non-polyglot beginner whom the author always tries to keep in mind. But it does not seem worth while to write for such a person, if the result is to be so sad a limitation of the actual theory presented. To treat linear difference equations without mentioning Birkhoff, or the gamma function without so much as a reference to Mellin—let alone his inversion formulae—these seem to us to diminish the importance of the already neglected difference calculus. The inclusion or exclusion of any particular topic must always rest with the author, who, undoubtedly has his reasons for the use of the

"Auswahlprinzip" in this sense. But the full reach of the modern developments of the subject in question might perhaps have been shown with more relief by some mention of the direct method of the calculus of variations; of the work of Schürer in mixed difference and differential equations. Perhaps a treatment of the Heaviside-Carson operational calculus could have been given by means of mixed equations in two variables, and might have added to the value of the work for physicists as well as engineers. Not only does one regret the omission of such really stimulating questions, but one feels at times that the stress laid on pure application is excessive. Even in interpolation formulae—the writer's strong point—one does not gain insight into the actual mechanism of the processes involved; whatever the value of theory may be for computation, some of us at least will always prefer such expositions as, for instance, are typified by the lecture of Fejér published in the *American Mathematical Monthly* (XLI, 1934, pp. 1-14).

It is to be hoped that the author sees fit, in some later edition to issue the first two hundred pages separately as an elementary text; and to do real justice to the "modern developments" as well as to his own undoubted talents in a second volume.

D. D. K.

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28. At a considerably higher level we find the contemporary mathematician who has still to learn the real meaning of "experimental verification," and who is habituated to treating the schemes of concepts in his brain as truer than fact, at odds with the modern biologist. Still constrained in the logical net, he shakes his head at the "unphilosophical" ease of the latter's mental movements. He objects to conclusions that are not final and exactly proved. He has not learned to rest in a provisional conclusion, and clings to the delusion that purely symbolical processes can win truth from the unknown. His symbolic processes never do win truth from the unknown, but he fancies that they justify an attitude of disapproval towards the pragmatical acceptances of practical science. But in the long run even the mathematicians will become scientific.

H. G. WELLS: *The Work, Wealth and Happiness of Mankind*, p. 69

SOLUTIONS TO QUESTIONS.

Question 1605.

(K. J. SANJANA AND K F. VAKIL):—In connection with Mr. R. Ramanujan's Question No. 1598 (*J. I. M. S.*, August 1931) prove the following two results.

(1) The length of the side BC of the pentagon ABCDE required to be constructed is given by the equation

$$BC \,.\, DG = 16992 \times (3 \,.\, 5 \;\; 7 \,.\, 13 \,.\, 47 \,.\, 59 \,.\, 61 \,.\, 83 \,.\, 317)^{\frac{1}{2}}.$$

(2) If P and Q are the lengths of the perpendiculars drawn from the point G respectively upon CD and BF, then $P/Q = 2867/437$.

Solution by V. A. Mahalingam, M. S Srinivasachari and the proposer.

B, C, D, F are cyclic. Hence by Ptolemy's Theorem,

$$BF.CD + BC.DF = BD.CF.$$

$\therefore \;\; BC.DF = BD.CF - BF.CD = CD(CF - BF)$

since $BD = CD$.

Also, since the triangles BGC and FGD are congruent,

$$\frac{DF}{BC} = \frac{GF}{BG} = \frac{83}{59} \quad i.e., \;\; DF = \frac{83}{59} \,.\, BC.$$

Substituting this for BF, and using the value of CD we have, after simplification,

$$BC^2 = \frac{59 \,.\, 2^{10} \,.\, 13 \,.\, 317, 5 \,.\, 3^5 \,.\, 59^2 \,.\, 7}{83 \,.\, 2867}$$

Also it is given that $DG^2 = 83^2 \,.\, 2867^2$

$\therefore \;\; BC \,.\, DG = 16992 \,.\, (3 \,.\, 5 \,.\, 7 \,.\, 13 \,.\, 47 \,.\, 59 \,.\, 61 \,.\, 83 \,.\, 317)^{\frac{1}{2}}.$

Lastly, the triangles GCD and GBF are similar and their altitudes are, therefore, as the corresponding sides. Hence

$$\frac{P}{Q} = \frac{CD}{BF} = \frac{GC}{BG} = \frac{59 \,.\, 2867}{19 \,.\, 23 \,.\, 59} = \frac{2867}{437}.$$

Question 1615.

(P. JAGANNATHAN) :—Prove that

$$\frac{2 \cdot (3n)!}{n!\,(n+1)!\,(n+2)!}$$

is an integer for all positive integral values of n.

Examine whether the factor 2 in the numerator may be omitted without affecting the result.

Solution by Hansraj Gupta.

Let p be any prime number. Then the number of times that p occurs as a factor of $q!$ is given by the expression

$$\sum_{k=1}^{\infty} [\, q/p^k \,] = [\, q/p \,] + [\, q/p^2 \,] + [\, q/p^3 \,] + \dots \dots,$$

where $[x]$ denotes the integral part of x. Hence p occurs as a factor of the denominator

$$\sum_{k=1}^{\infty} \{ [\, n/p^k \,] + [\, (n+1)/p^k \,] + [\, (n+2)/p^k \,] \}$$

times. It occurs as a factor of the numerator $\sum_{k=1}^{\infty} [\, 3n/p^k]$ times.

Now

$$[\, n/m \,] + [\, (n+1)/m \,] + [\, (n+2)/m \,] \leqslant [\, 3n/m \,]$$

for all values of $m \geqslant 3$; for let $n = am + n_1$, where $n_1 < m$. Then $[\, n/m \,] + [\, (n+1)/m \,] + [\, (n+2)/m \,] = 3a,\ 3a+1$, or $3a+2$ according as $n_1 <, =$ or $> m - 2$; and

$$\left[\frac{3n}{m}\right] = \left[\frac{3am + 3n_1}{m}\right] \geqslant 3a, \qquad \text{if } n_1 < m-2,$$

$$= \left[\frac{3am + m + (2m-6)}{m}\right] \geqslant 3a+1, \text{ if } n_1 = m-2,\ m \geqslant 2,$$

$$= \left[\frac{3am + 2m + (m-3)}{m}\right] = 3a+2, \text{ if } n_1 = m-1,\ m \geqslant 3.$$

Also

$$[\, n/2 \,] + [\, (n+1)/2] + [\, (n+2)/2 \,] = 1 + [\, 3n/2 \,]$$

for, if $n = 2a$, then

$$[n/2] + [(n+1)]/2] + [(n+2)/2] = 3a+1 = 1 + [3n/2] \quad \dots \text{(A)}$$

while if $n = 2a+1$, it is $\quad 3a+2 = 1+[3n/2]$.

These lemmas go to prove the given result.

From (A) it is clear that the factor 2 is necessary in the numerator. If, however, $n \geqslant 3$, then for some value of k, $2^{k-1} \leqslant n + 2 < 2^k$. Hence, $[n/2k] + [(n+1)/2k] + [(n+2)/2k] = 0$, while $[3n/2k] = 1$. So in this case the factor 2 is not necessary in the numerator.

Also solved by S. Minakshisundaram and the proposer.

Question 1639.

(K. SATYANARAYANA) :—Show that the locus of the centre of a conic with axes in given directions, and for which a given triangle is self-conjugate, is a rectangular hyperbola passing through the vertices whose asymptotes are parallel to the axes of the conics.

Solution by Altshiller Court, Hukam Chand, S. Pichai, V. Ramaswamy Aiyar, R. Ranganatha Rao, K. Rangaswami, H. S. Subba Rao and G. Sankara Iyer.

If O be the centre of such a conic and ABC the given triangle and P, Q the points at infinity in the given directions, then ABC, OPQ are both self-polar with regard to the same conic. Hence all the six points lie on a conic, *i.e.*, O lies on the conic determined by ABCPQ which is the rectangular hyperbola described in the question.

Analytical Solution by Hansraj Gupta, Hariharan, Hukam Chand, S. P. Ranganathachar, V. Ranganathan and Vidyachandra.

Question 1642.

(C. N. SRINIVASAIENGAR) :—At any point of a non-torsal generator of a scroll, the scroll and the limiting conicoid through the generator and two generators consecutive to it have the same principal radii of curvature. Also if any two surfaces touch along a generator, their Gaussian curvatures at any point along it are equal.

Solutions (1) by the Proposer; (2) by K. Rangaswami.

(1) For any surface, the two asymptotic lines through any point have the same torsion numerically, *viz.*, $\sqrt{K}$ where K is the Gaussian curvature. The proof of this result, called Enneper's formula, depends upon the property that the binormals to the curve at any poin coincides with the normal to the surface thereat. Hence, in the case of a generator of a surface (not necessarily a scroll), if we define the binormal as the normal of the surface, the torsion of the generator at any point will be given by the same formula. (There will be no difficulty in seeing that the Serret-Frenet formulae continue to hold for the straight line with this definition of the binormal). The second part of the question now follows.

For a given generator of a scroll, there are ∞^3 conicoids which touch the scroll along the generator. For the particular conicoid considered, both the inflexional tangents at any point on the generator are the same as for the scroll. Now, the semi-angle between the inflexional tangents is given by $\tan^{-1}\sqrt{-\rho_2/\rho_1}$. Therefore $\rho_1\rho_2$ as well as ρ_2/ρ_1 are the same for both the surfaces. Hence the result.

(2) The first part is proved in Bell's *Co-ordinate Geometry*, p. 346.

At any point of a generator g distant x from the central point, the value of the Gaussian curvature is $-p^2/(p^2+x^2)^2$ where p is the parameter distribution for the generator (*vide* Bell: *Co. Geom.*, p. 348). Also if two surfaces touch along a generator, the central point and the parameter of distribution of the generator are the same for both the surfaces. Hence their Gaussian curvatures along g are equal.

Also solved by P. Jagannathan, V. T. Srinivasan.

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* BELL: *Co-ordinate Geometry*, § 245; Weatherburn: *Differential Geometry* 38. Also, *vide*, J. I. M. S., Vol. III, p. 86.

ANNOUNCEMENTS AND NEWS.

Readers are invited to contribute to the value of this section by sending suitable news items of interest.

In accordance with our declared policy of publishing from time to time portraits of Mathematicians who have rendered conspicuous service to the stndent world or have made noteworthy contributions to the subject, we have great pleasure in including in this issue a portrait of Prof. G. N. Watson, Sc.D., F.R.S. There are few students of higher mathematics in Indian Universities to whom Prof. Watson's classical joint work on *Modern Analysis* has not been a source of help and inspiration; while his encyclopaedic treatise on ***Bessel Functions*** is a mine of information to serious students of the subject.

The Managing Committee of the Society has passed a resolution recording its appreication of the valuable services rendered by Mr. S. R. Ranganathan as Hon. Treasurer of the Society for nearly six years.

The Committee has approved of the following concessions regarding Life-composition fees where the membership has been continuous and long-standing; namely, Rs. 100, Rs. 75, Rs. 50 and Rs. 25 respectively for those whose membership extends continuously over 10, 15, 20 and 25 years. Members of 30 years' standing are exempt from all fees. The normal life composition fee is Rs. 150.

These concessions are made in deference to the wishes of the Members of the Society as expressed at the general body meeting held at the Jubilee Conference.

The Society acknowledges with gratitude the receipt of financial help from the Madras, Bombay and Annamalai Universities to the extent mentioned below:

		Year of receipt.
Madras University	Rs. 350	1933
Bombay University	,, 200	1934
Annamalai University	,, 100	1934

Dr. Alfred Moessner in Nurnberg, Germany, Nordring 38, wishes to announce that he would gladly enter into correspondence with all readers of the STUDENT interested in the Theory of Numbers.

In recognition of his long-continued and important contributions to the cause of American Mathematics the members of the Mathematical Association of America have voted unanimonsly to suspend the bye-laws and to elect Professor H. E. Slaught Honorary President of the Association for life.

Sir James Jeans has been elected President of the British Association for the year. The Presidential address is to be delivered at Aberdeen in September.

The tenth International Congress of Actuaries will be held at Rome between the 4th and the 10th of May, 1934.

Sir C. V. Raman has been invited to deliver the inangural address at the first International Congress of Radiobiology at Venice in September 1934.

The Cambridge University has awarded its Smiths prizes to K. Mitchell, Esq., B.A., of Peterhouse and T. Ward, Esq., of Emmanuel College, and its Raleigh Prizes to M. S. Bartlett, Esq., B.A., of Queen's College and C. G. Pendse, Esq., B.A., of Downing College. Mr. C. G. Pendse is the son of Prof. G. G. Pendse of Baroda College, a member of the Society, and his essay was on the Theory of Saturn's Rings. We offer our felicitations to the prize-winners.

Prof. G D. Birkhoff of the Harvard University has been awarded a prize of 1000 lire donated by Pope Pius XI for the best book on "system for the solution of differential equations."

At the next International Conference of Mathematicians to be held In Oslo (formerly Chiristiania), Norway in 1936, two gold medals are to be awarded to out-standing mathematicians. The foundation is due to Dr. J. C. Fields, President of the Toronto Mathematical Congress (1924) and is financed out of the proceeds remaining from the sale of the *Proceedings* of the Toronto Congress.

The Jeypore Vikram Deo College of Science and Technology and the University College of Arts were opened by the Chancellor, Andhra University, in December 1933.

Mr. Gian Chand, M.A., has been appointed Lectnrer in Mathematics, Ramja's College (Delhi University).

K. Nagabhushanam, Esq., research scholar in the Andhra University, has been appointed Lecturer in Mathematics in the same University.

Sir Thomas Muir died on 21st March 1934. By a deed of gift he has bequeathed his library to the South African state

The Indian Mathematical Society

Statement of Accounts for the Year 1933.

RECEIPTS.

				Rs.	A.	P.
Balance from 1932						
Savings Bank a/c ...	2,916	8	11			
Current Account ...	147	2	6			
Fixed Deposit ...	3,800	0	0			
				6,863	11	5
Subscription from Members ...				1,189	0	0
Life Subscription (from Prof. Panday) ...				75	0	0
Subscription for Journal ...				157	3	0
Grant in Aid (from the Madras University)				350	0	0
Interest on investments ...				196	6	5
Silver Jubilee Conference ...				500	0	0
Miscellaneous ...				32	12	6
Total ...				9,364	1	4

EXPENDITURE.

				Rs.	A.	P.
Ordinary Working Expenses ...				210	7	3
Books and Periodicals ...				765	14	0
Journal and other Printing ...				655	3	0
Library Expenses ...				450	0	0
Closing Balance :—						
Fixed Deposit ...	5,600	0	0			
Savings Bank a/c ...	1,540	2	10			
Current Account ...	142	6	3			
				7,282	9	1
Total ...				9,364	1	4

(Sd.) L. N. SUBRAMANIAM,
Hony. Treasurer.

Audited and found correct.

(Sd.) G. A. SRINIVASAN,
Hony. Auditor.

QUESTIONS FOR SOLUTION.

Proposers of Questions are requested to send their own Solutions along with their Questions.

1668. (ALFRED MOESSNER, Nurnberg, Germany):—

(i) We have

$$17^n + 17^n = (-3)^n + 5^n + 12^n + 20^n \qquad (n = 1, 2, 3).$$

Is the equation $A_1^n + A_2^{2n} = B_1^n + B_2^n + B_3^n$ $(n = 1, 2, 3)$ also solvable when $A_1 \neq A_2$? What is the general solution?

(ii) We have $7^n + 7^n = 3^n + 5^n + 8^n$ $(n = 2, 4)$.

Is the equation $E_1^n + E_2^n = F_1^n + F_2^n + F_3^n$ $(n = 2, 4)$ also solvable for $E_1 \neq E_2$?

1669. (HANSRAJ GUPTA):—If $|x| < 1$, prove that

$$1/(1-x^2)(1-x^3)(1-x^4)\ldots = 1 + \sum_{s=1}^{\infty} x^{2s}/((1-x)(1-x^2)\ldots(1-x^s)).$$

1670. (B. RAMAMURTI):—If $f_r(x)$, $r = 1, 2, 3, 4$, be four linearly independent polynomials of the same degree, and J_{rs} denotes the Jacobian of $f_r(x)$ and $f_s(x)$ prove that

$$\Sigma\, [J_{12}(x)\, J_{34}(y) + J_{12}(y)\, J_{34}(x)]$$

has the factor $(x - y)^4$, the summation corresponding to the permutations (12) (34), (23) (14) and (31) (24).

Hence, or otherwise, prove that if r_1, r_2, r_3, r_4 be four distinct integers, the equation $\Sigma\,(x^{r_1+r_2-1} + x^{r_3+r_4-1})(r_1-r_2)(r_3-r_4) = 0$ has unity as a quadruple root.

1671. (NATHAN ALTSHILLER COURT, Oklahama, U.S.A.):—Consider the variable line on which two fixed spheres determine two harmonic segments. The locus of the mid. points of these segments is a sphere.

1672. (V. RAMASWAMI AIYAR):—ABCD is a quadrilateral inscribed in a cirle, centre O. BC, AD intersect at P. If X, Y are the circum-centres of the triangles ABP, CDP, show that the circum-radii of these triangles are equal to OY, OX.

1673. (V. RAMASWAMI AIYAR) :—A triangle $\alpha\beta\gamma$ moves rigidly and without rotation, so as to be constantly in perspective with a given triangle $A_0B_0C_0$. Prove that there is an infinity of fixed triangles ABC with respect to each of which the moving triangle $\alpha\beta\gamma$ is constantly in perspective. Prove further that the locus of the centre of perspective, corresponding to different triangles ABC, forms a system of concentric, similar and similarly situated conics [briefly *co-asymptotic conics*].

1674. (T. R. RAGHAVA SASTRY) :—O is a fixed point in the plane of a triangle ABC. The reciprocal of ABC with respect to O is the triangle PQR. It is stated in Question 1631 that the centre of perspective of the triangles ABC and PQR for different radii of reciprocation traces out the rectangular hyperbola through ABCO.

Show further that the axis of perspective of the two triangles envelops a parabola inscribed in ABC and having the line joining O to the ortho-centre of ABC as directrix.

1675. (A. RANGANATHA RAO) :—A point P and a line l are pole and polar with respect to each of two conics S, S′. If *one* chord AB of S′, which is a tangent to S be such that PA, PB are conjugate lines with respect to S, then the same is true of *all* chords of S′ which are tangents to S.

1676. (A. A. KRISHNASWAMY AYYANGAR) :—Show that there is a unique solution to the letters A, E, G, S and the missing digits, which will make the following multiplication true. No two different letters represent the same digit.

$$\begin{array}{cccccccc}
 & & & E & * & * & * & S \\
 & & & & & A & G & E \\
\hline
 & & * & * & * & E & G & * \\
 & & * & * & G & * & E & \\
G & E & * & * & * & A & & \\
\hline
G & A & * & * & E & * & * & *
\end{array}$$

RECENT PUBLICATIONS.

We are opening this column to bring important new publications in Mathematics to the notice of our readers from time to time.*

BAKER Principles of Geometry, Vols. 5 and 6 ; Cambridge University Press.

BERNSTEIN Lecons sur les progres recents de la theorie des series de Dirichlet (Preface by HADAMARD), Collection Borel, Gauthier Villars, Paris.

BIRKHOFF Aesthetic Measure, pp. 226, Harvard University Press. $7.5.

CARTAN Sur la structure des groupes de transformations finis et, continus ; Vuibert, Paris.

EECKE Pappus d'Alexandrie, Tr from the Greek, De Brouwer, Paris and Bruges.

EINSTEIN Les fundaments de la theorie de la relativite generale ; Theorie unitaire de la gravite et de l'electricite ; Sur la structure cosmologique, Tr. by Solovine, Herman, Paris,

EULER Opera Omnia, Vol. 16, Teubner.

HARDY, LITTLEWOOD and POLYA Inequalities. Cambridge, 18s.

HILBERT and BERNAYS Grundlagen der Mathematik (Grundlagen d. Math. Wiss. No. 40 Springer.

HURWITZ Werke, Basel Buchhauser.

INGE God and the Astronomers, Longmans.

MACPIKE Halley, correspondence and papers, Oxford, 21s.

MERRIL Mathematical excursions, The Norwood Press.

MITCHELL Elements of mathematical Analysis, 600 pp. Macmillan.

MORLEY Inversive Geometry, G. Bell and Sons, 18s.

PHILLIPS Vector Analysis, John Wiley and Sons, N. Y.

SOMERVILE Analytical Geometry of three dimensions, Cambridge 18s.

THOMAS Differential invariants of generalised spaces, Camb. 21s.

WALKER Conjugate functions for beginners, Oxford.

WHITEHEAD Adventures of ideas, Cambridge, 12s. 6d.

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* Publishers of mathematical books are requested to give the Editor details regarding their publications with a view to insertion in this column, if space permits. Details regarding the contents of the books can only be accepted as advertisements.

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EXPOSITORY REMARKS ON THE THEORY OF GROUPS.*

BY G. A. MILLER,
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The technical term *group* appears in the mathematical literature with widely different meanings. For instance, the explicit definitions of this term which appear in the *Encyclopédie des Sciences Mathématiques,* tome 1, volume 1, page 576, and tome 1, volume 2, page 243, respectively, differ widely. The latter of these definitions is too general, to serve as a basis for any extensive theory in the present state of mathematical development. In fact, the former is also too general, if it is assumed that it is supposed to apply to groups of infinite order as seems to be implied by the first sentence which follows this definition At any rate, it is the definition of a semi-group, *cf.* Pascal's *Repertorium der höheren Mathematik,* volume 1, (1910), page 191. The same unsatisfactory definition appears in the German edition of the said Encyclopaedia, Volume 1, page 217, where it is stated that it applies also to groups of infinite order, and hence the student of mathematics cannot find satisfactory information relating to the definition of the term group by consulting our best modern mathematical encyclopaedias.

* [*The Mathematics Student* hopes to publish, from time to time, expository papers dealing with developments in important topics in Modern Mathematics. A paper by Dr. Chowla reviewing the progress in the Theory of Numbers appeared in Vol. I. pages 41—48. The Editor wishes to thank Prof. Miller for having kindly responded to a similar request for a review article on Group Theory. —ED.]

One of the simplest definitions of the technical term *group*, which applies to the groups of infinite order as well as to those of finite order may be stated as follows : A set of distinct elements $s_1, s_2, s_3, \ldots \ldots$ constitutes a group if it satisfies the conditions that if any two of the three symbols in the equation $xy = z$ are replaced by the same element or by two different elements of this set, the equation will be satisfied by one and only element of the set. Moreover, the elements obey the associative law when they are combined successively two at a time or one with its powers. This definition appears in the treatise on *Finite Groups* by Miller, Blichfeldt, Dickson, 1916, page 52. An equivalent definition is found in H. Weber's *Kleines Lehrbuch der Algebra*, 1912, page 181, but a redundant condition is given here as the first condition and hence Weber's definition is unnecessarily long and therefore undesirable.

The fact that only two elements of a group are supposed to be combined at one time has not been sufficiently emphasized in the definitions of the technical term *group*. To see that this emphasis is desirable, it is only necessary to note that if in a given group whose order exceeds 2 we multiply a product of two or more elements by the number of these elements, and this number is not constant, the result is not a group. Even the common positive rational numbers when they are combined by multiplication do not constitute a group if the products, consisting of different numbers of factors, are always multiplied by the number of these factors. It should be noted that this method of combining the rational numbers is a perfectly definite operation but it is not a group operation. It is desirable to distinguish clearly in the mathematical developments between group operations and non-group operations. *Cf.* G. A. Miller, *Proceedings of the National Academy of Sciences of the United States of America*, Volume 18 (1932), page 598.

Another feature of the definition of a group, which seems to deserve more emphasis than it has received in the published definitions of this term, is the meaning of the phrases "any two," "every two," or "every

pair." In the former of the two definitions to which we referred near the beginning of this article, the first of these phrases is used. The exact meaning of these phrases is important in group theory but it is not always correctly understood by the beginner. If there are n distinct objects, the number of the possible pairs is $n(n-1)/2$ or $n(n+1)/2$ according as no repetitions or as repetitions are considered. The former represents the number of simple pairs while the latter represents the number of complete pairs. In elementary mathematics the student often becomes familiar only with the former types of pairs, but in the definitions of the technical term *group*, the latter are commonly implied even when this is not explicitly stated. In the volume noted at the close of the preceding paragraph, page 100, all the groups are determined in which it is possible to select a set of generating operators in such a way that they include every simple pair thereof but not every complete pair, and hence the set does not constitute a group. This exhibits the great desirability of an explicit statement relating to this point in the definitions of a group.

Many of the difficulties encountered by the student of mathematics are inherent in the subject and these can be overcome only by serious meditations; but many others are due to a lack of clearness in the presentations, and these should be banished from mathematical literature as rapidly as possible. The literature of group theory is still in need of many such banishments and the present article aims to be effective in this direction. The equational definition of the technical term *group* which is found in the second paragraph of this article should replace many of the longer definitions of this term. According to this definition, the natural numbers do not constitute a group when they are combined by multiplication as may be seen by substituting in the given equation a prime number for z and a different prime number for x. Notwithstanding this obvious contradiction, one of the most widespread errors relating to our subject is the statement that the natural numbers constitute a group when they are combined by multiplication. In the first edition of Volume 2 of H. Weber's *Lehrbuch der Algebra*, 1896,

page 54, it is stated that this even constitutes the most important example of an abelian group, but this statement was fortunately corrected in the second edition of this volume.

Even the cyclic group of infinite order has been treated quite differently by different writers. It is sometimes said that this group has two independent generators, s_1, s_2 which satisfy the equation $s_1 s_2 = 1$. *Cf.* Pascal's *Repertorium der hoheren Mathematik,* Volume 1 (1910), page 193. If this is done, it results that different powers of the same operator are independent operators. In particular, if we consider the cyclic group of infinite order composed of the powers of 2, it would result that 2 and $\frac{1}{2}$ are its independent generators even though both of these numbers are powers of 2. While there are some advantages in this point of view, it must be regarded as a convention which establishes an unnecessarily wide difference between the treatment of the cyclic groups of finite and of infinite order. It would seem wiser to say that a cyclic group of infinite order as well as a cyclic group of finite order has a single generator and that each of its operators is a power of this generator, as is done, for instance, by H. Weyl, *Gruppentheorie und Quantenmechanik*, 1931, page 105.

The definitions of the abstract groups of finite order were usually based on the properties of permutation groups since it was found early in the development of group theory that every such abstract group can be represented as a permutation group. On the contrary, there has not been such a useful guiding principle in the formulation of the definitions of abstract groups of infinite order and hence a greater lack of uniformity may be observed in the latter definitions than in the former. Even such an eminent authority on groups of an infinite order as S. Lie asserted that there are groups which contain neither the identity nor the inverses of their operators; *Theorie der Transformationsgruppen*, Volume 1 (1888), page 163. Such assertions merely imply that S. Lie included here under the term group what other writers call semi-groups. Although such variations in the use of the same term impose unnecessary difficulties on the beginner, they can scarcely be avoided in the developments of a subject in which a large number take an active part.

Suppose that G represents a finite group of order g and that s represents any one of its operators except the identity. The successive powers of s are also operators of G and hence not more than g of them can be distinct. If s^m is the first power of s which is equal to a lower power thereof, then s^{m-1} is the identity of G. If m is less than g, then G contains a proper sub-group of order m. If m is equal to g but composite, then a power of s generates a proper sub-group of G. Hence it results that every group, except the groups of prime orders, contains a proper sub-group and hence its operators can be arranged in the form of a rectangle in which this proper sub-group is the first row. Such an arrangement appears in a letter from P. Abbati to P. Ruffine, dated September 30, 1802, and hence it is one of the earliest developments in the theory of groups and also one of the most important since it is useful not only in establishing the fact that the order of a group is divisible by the order of every one of its sub-groups, but also to prove that the operators of every other sub-group of G are equally distributed among the rows which involve at least one such operator.

From the preceding paragraph, it results that it was easy to prove that every group whose order is not a prime number, contains at least one proper sub-group. It was much more difficult to find, in general, the order of such a sub-group. The first step in this direction was taken by E. Galois, who asserted without giving any proof, as far as we know, that whenever g is divisible by a prime number p, then G contains a sub-group of order p. The earliest known proof of this fundamental fact appears in A. L. Cauchy's *Exercices d' Analyse*, Volume 3, 1844. L. Sylow later (1872) embodied it in what is now commonly known as Sylow's theorem, which establishes the fact that if p^m is the highest power of p which divides g, then G contains at least one sub-group of order p^m. If it contains more than one such sub-group, the number of these sub-groups is of the form $1 + kp$, and they are all conjugate under G. This theorem establishes the existence of some but not necessarily all the sub-groups which appear in a group whose order is given.

Sylow's theorem gives some information in regard to the number of the sub-groups of order p^m which can appear, but it gives very little

information in regard to the number of the operators whose orders are powers of p which appear in G. It is easy to prove that every such operator appears in at least one of the Sylow sub-groups of G and that G contains at least p^{m+1} operators whose orders are powers of p whenever it contains more than one Sylow sub-group or order p^m; *Cf.* Miller, Blichfeldt, Dickson, *Finite Groups,* 1916, page 80. No one knows how many theorems on group theory are now in the mathematical literature. In the second edition (1927) of his excellent *Theorie der Gruppen von endlicher Ordnung*, A. Speiser numbers his theorems from 1 to 217; but he naturally does not include nearly all of the known theorems on this subject, and various theorems can often be combined into a more general one. It would not be difficult to make a list of more than 500 theorems relating to groups of finite order alone, and there is still room for many new and interesting ones.

In support of this last assertion, we proceed to develop here a few new theorems relating to an elementary question in group theory, *viz.*, the number of operators whose orders are a power of a prime number p in a group whose order is divisible by p^m but by no higher power of p. One of the fundamental questions in this connection is whether there are groups which have a set of Sylow sub-groups such that none of these sub-groups involves an operator which does not also appear in another such sub-group. To prove a theorem according to which this question can be answered in the affirmative, it is desirable first to note the theorem that the number of the Sylow sub-groups of order 2^m in any given symmetric group of degree n is $n!/2^m$. This theorem results directly from the fact that when n is of the form 2^k, then the Sylow sub-group of order 2^m contained in the symmetric group of degree n is the direct product of two Sylow sub-groups contained in the symmetric group of degree $n/2$ extended by an operator of order 2 which merely interchanges these two Sylow sub-groups. Hence such a Sylow sub-group contains one and only one substitution of order 2 which is commutative with every one of its substitutions.

From the method used in the preceding paragraph to construct a

Sylow sub-group of order 2^m contained in the symmetric group of degree $2k$, it is easy to see how a Sylow sub-group of order 2^{m-1} of the corresponding alternating group can be constructed and that when $k > 2$ there is one and only one substitution of order 2 which is commutative with every substitution of this alternating group and contained therein. Such a substitution of order 2 appears in more than one of the Sylow sub-groups of order 2^{m-1} contained in this alternating group; but if two Sylow sub-groups of this order in this alternating group involve different invariant substitutions of order 2 they are necessarily distinct. Greater details in regard to these developments appear in an article by the present writer and published under the title "On the transitive substitution groups whose order is a power of a prime number," *American Journal of Mathematics*, Volume 23 (1901), page 173.

It is now easy to prove that when $k > 3$ is even, the Sylow sub-groups whose orders are a power of 2 in the alternating group of degree 2^k have the property that every substitution contained in one of them appears also in at least one of the others. To prove this theorem it is only necessary to observe that every such substitution is commutative with more than one substitution of order 2 and of degree 2^k which appears in this alternating group. If a positive substitution whose degree does not exceed 2^k is commutative with only one substitution of order $2k$ on these 2^k letters, all its cycles have different orders and hence they omit at least four letters. As different substitutions of order 2 can be formed on the omitted letters, the theorem in question has been proved. With the increase of the publications on group theory, the beginner in this field is compelled to familiarize himself with a larger number of theorems relating thereto before he can be sure that his discoveries are new; but he also has the advantage that the border line between the known and the unknown is continually getting longer, and hence there is a larger variety of problems which await solution.

The development of group theory during the nineteenth century gained momentum as the century advanced and represents one of the

most prominent features of the mathematical developments of this century. In reviewing these developments, J. Pierpont made the following remark: " In resume, we may thus say that the group concept, hardly noticeable at the beginning of the century, has at its close become one of the fundamental and most fruitful notions in the whole range of our science." *Bulletin of the American Mathematical Society*, Volume 11 (1904), page 144. At the beginning of the century it was regarded merely as a useful concept in the development of a general theory of algebraic equations while at the end thereof it had permeated so many mathematical theories that H. Poincare was led to remark as follows: " The theory of groups is so to say entire mathematics divested of its matter and reduced to pure form, " *Acta Mathematica*, Volume 38 (1921), page 145. A recent very useful brief summary of the most fundamental developments in this field may be found in an Italian work entitled *Enciclopedia delle Matematiche Elementari*, Volume 1, part 2 (1932), pages 19—69. This extensive article was written by L. Berzolari and is especially interesting because it appears in a work on elementary mathematics and consequently it emphasizes the fundamental ideas of our subject.

GLEANINGS.

24. We reject the thesis of the categorical finiteness of man, both in the atheistic form of obdurate finiteness which is so alluringly represented to-day in Germany by the Freiburg philosopher Heidegger, and in the theistic, specifically Lutheran-Protestant form, where it serves as a back-ground for the violent drama of contrition, revelation and grace. On the contrary, mind is freedom within the limitations of existence; it is open towards the infinite. Indeed, God as the completed infinite cannot and will not be comprehended by it; neither can God penetrate into man by revelation, nor man penetrate to him by mystical perception. The completed infinite we can only represent in symbols. From this relationship every creative act of man receives its deep conservation and dignity. But only in mathematics and physics, as far as I can see, has symbolical-theoretical construction acquired sufficient solidity to be convincing for every one whose minds open to these sciences.

DR. HERMAN WEYL: *The open World.*

25. MEMORISING LOG π

The logarithm of *pi* to the base ten is often used in sceintific problems. To memorize this value to 20 decimal places, memorize the following rhyme Each word represents the digit of the number which is determined by the number of letters in the word. " No " represents Zero; $\log_{10} \pi = 0.49714987\ 2694133\ 85435$.

No task, therefore, becomes a bore
Following pleasant Science
It simply increases what I use, and
Supplies might, like the lion's,.

Earl C. Rex, M. S. in *The Scientific American*, May 1934.

SOME RESULTS CONNECTED WITH TRIANGLES IN PERSPECTIVE.*

BY K. SATYANARAYANA,
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In what follows, O is the centre and XYZ the axis of perspective of two triangles ABC, A'B'C'; and A", B", C" are points on OA, OB, OC such that OA" = A'A, etc.; (F) (F'), (F") are the circum-conics of OABC, OA'B'C', OA"B"C"; and Γ is the conic for which the triangles ABC, A'B'C', are reciprocal.

It is proposed to present in this paper two types of results. The first type developed in paras 1 to 4 is in respect of certain lines (designated *polar lines* and *polar axes*) connected with the triangles and their relations to Γ and to the *s. s. p.*† members Σ, Σ', Σ'' of (F), (F'), (F"); the condition that each of two triangles in perspective shall be self conjugate for the same conic; and the condition that Γ be a parabola. The second type outlined in the rest of the paper relates to the theory of the rigid relative displacement of the triangles without disburbance of perspectivity.

A detailed discussion of the results given in Question 1648 by Mr. Ramaswamy Aiyar will be found in the last para.

1. On certain lines connected with the triangles.

Let a transversal cut OA, OB, OC in L, M, N; and let the intersections of YN, ZM; ZL, XN; XM, YL be respectively H_1, H_2, H_3. By projecting XYZ to infinity and O into the common ortho-centre of *abc* and *a'b'c'*‡, it is easily seen that h_1, h_2, h_3 are respectively the ortho-centres of *omn*, *onl*, *olm* and lie on the perpendicular from o on *lmn* Obviously to all lines parallel to *lmn* corresponds the same line $h_1h_2h_3$. Also the parallel to *lmn* through o and the line $h_1h_2h_3$ are conjugate rays of the involution pencil *oa*, *ox*; *ob*, *oy*; *oc*, *oz*. Hence

To *every* ray LMN of a pencil with its vertex V on the axis of perspective corresponds the same line $OH_1H_2H_3$. We shall call this line the *polar line* ‖ *of perspective* of V. ... (1.1)

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* I am indebted to Mr. V. Ramaswamy Aiyar whose Questions 1648, 1650, 1651, 1660, *Math. Stud.*, Vol. I have helped in this investigation.

† '*s. s. p.*' is used as an abbreviation for 'similar and similarly placed.

‡ Small letters indicate the projections.

‖ For the significance of this word, *vide* result (4.11).

OV and the polar line of V are conjugate rays of the involution pencil of perspective, *viz.*, OA, OX; OB, OY; OC, OZ. ... (1.2)

The polar line of the point at infinity on XYZ may in particular be called the *polar axis* of perspective of the triangles. It corresponds to all lines LMN parallel to XYZ as well as to the line at infinity, and cuts XYZ in the centre of the involution range formed on XYZ by the involution pencil of perspective. ... (1.31)

By considering the polar axis as corresponding to the line at infinity, it is easy to see that :—

If O is the ortho-centre of ABC, then the polar axis contains the ortho-centres of the triangles AYZ, BZX, CXY. ... (1.32)

2. Certain lemmas on *s. s. p.* conics.

The following lemmas on *s. s p.* conics may be proved directly or by projecting them into circles. We take Σ, Σ' to be two *s. s. p.* conics cutting in O, O′.

If AOA′, PO′P′ be two chords, AP is parallel to A′P′; conversely if AOA′ be a chord and AP, A′P′ be parallel, then PO′P′ is a straight line. ... (2.1)

OAA′, OBB′ being any two chords, if AB is parallel to A′B′, then Σ, Σ' touch at O. ... (2.2)

If Σ, Σ' touch at O and OAA′, OBB′, OCC′ be any three lines cutting Σ, Σ' respectively in A, B, C; A′, B′, C′, then the triangles ABC, A′B′C′ are *s. s. p.*, O being the centre of similitude. ... (2.3)

If on *any* chord OAA′, a point P be taken such that OP = AA′, the locus P is a *s. s. p.* conic through O touching the common chord OO′. ... (2.4)

Its semi-diameter through O equals the distance between the centres of Σ and Σ'.

3. *s. s. p.* members of (F), (F′), (F″).

If the triangles ABC, A′B′C′ be *s s p*, then to any conic of (F) corresponds a unique conic of (F′) *s. s. p*, to the former and the two touch each other. ... ... (3.1)

For taking any member Σ of (F), consider the *s. s. p.* conic through O, A′, B′. This touches Σ by (2.2) and passes through C by (2.3).

If the triangles ABC, A′B′C′ are not similarly placed, there exists (except when O is at infinity) a unique member Σ of (F) and a unique member Σ' of (F′) which are *s. s. p.* ... (3.2)

For, the pencils of conics through OABC and OA′B′C′ cut the line at infinity in pairs belonging to two involutions, and we know there is always one pair common to two involutions. The conics through these points fulfil the required conditions. If O is at infinity, it is easy to show that to every member of (F) corresponds a *s. s. p.* member of (F′).

It therefore follows that, when O is not at infinity and the triangles ABC, A′B′C′ are not similarly placed, if two *s. s. p.* members of (F), (F′) be discovered, these should be Σ, Σ'. For example:

If OABC, OA′B′C′ be cyclic quadrilaterals, Σ, Σ' are the circum-circles: ... (3.21)

If a single pair of corresponding sides, say AB and A′B′, be parallel, then Σ, Σ' are respectively AB, OC; A′B′, OC′. ... (3.22)

If A, B, C be collinear, Σ consists of ABC and another line through O. So, if A′, B′, C′ be not collinear but a side, say A′B′, is parallel to ABC, then Σ, Σ' consist respectively of ABC, OC and A′B′, OC ... (3.231)

If A′, B′, C′ be also collinear (but not parallel to ABC), Σ consists of ABC and a parallel through O to A′B′C′ and Σ' consists of A′B′C′ and a parallel through O to ABC. ... (3.232)

From (2.4) it follows that when Σ, Σ' are the conics under reference in that result, a conic Σ'' *s. s. p.* to Σ, Σ' belongs to (F″) Hence from (3.2), it follows that:—

If the triangles ABC, A′B′C′ be not similarly placed, then (F), (F′), (F″) possess respectively three unique s. s. p. conics Σ, Σ', Σ''. .... (3.3)

It may be mentioned that the triangle A″B″C″ cannot be similarly placed to either of the triangles ABC, A′B′C′.

The polar axis of the triangles ABC, A′B′C′ is the common chord of Σ, Σ' *and touches* Σ''. ... (3.4)

To prove this, project O into the common ortho-centre of the triangles abc, $a'b'c'$. Then the projections of Σ, Σ' are rectangular hyperbolas having the projection of the line at infinity as a common chord. Hence, since the common points of two rectangular hyperbolas form an ortho-centric tetrad, the other common chord through the projection of O must be perpendicular to the projection of the line at infinity. So in the original figure the common chord is the polar axis. From (2.4) it follows that this polar axis touches Σ''. Hence

If Σ, Σ' be two *s. s, p.* conics through O and if OAA′, OBB′, OCC′ be *any* three chords, the triangles ABC, A′B C′ have always the same polar axis, namely, the common chord of Σ, Σ'. ... (3.5)

4. The Conic Γ.

Since the involution pencil of perspective is the pencil of conjugate lines of Γ, it follows that

V and its polar line of perspective are pole and polar for Γ and the polar axis of perspective is the diameter of Γ conjugate to the axis of perspective. ... (4.11)

If the involution pencil of perspective be orthogonal, O and XYZ being then a focus and the corresponding directrix, the polar axis is a principal axis of Γ. ... (4.12)

The Γ's of all pairs of triangles in perspective with their corresponding vertices on three fixed straight lines and with their corresponding sides concurrent at three fixed points have double contact. ... (4.2)

This follows since the involution of perspective in each case is the same. The double contact is at the double points of the involution range formed on XYZ.

Let XYZ cut OA, OB, OC in α, β, γ. If A_1, A'_1 belong to the involution (Oα, AA′), ZA , ZA_1' cut OB in B_1, B_1', belonging to the involution (Oβ, BB′) and YA_1, YA'_1 cut OC in C_1, C'_1 belonging to the involution (Oγ, CC′). Project O into the common ortho-centre of the triangles abc, $a'b'c'$. Then o is the ortho-centre of the triangles $a_1b_1c_1$, $a'_1b'_1c'_1$. Also o is the centre of the projections of the involution ranges so that $oa.oa' = oa_1.oa'_1$. Hence $a_1b_1c_1$, $a_1'b_1'c_1'$ are reciprocal for the circle for which abc, $a'b'c'$ are reciprocal. Hence pairs of triangles like $A_1B_1C_1$, $A'_1B'_1C'_1$ are also reciprocal for Γ. ... (4.31)

It is easy to prove that, if L, L′; M, M′; N, N′ be the respective double points of the involutions on OA, OB, OC, there are two triangles LMN, L′M′N′ each self-conjugate for Γ. ... (4.32)

The solution of the proper vertices of these triangles is to be made as in (4.31). From the above, incidentally it follows that :

The necessary and sufficient condition that two triangles in perspective shall be each self-conjugate for the same conic is that each pair of corresponding vertices should be separated harmonically by the centre and axis of perspective. ... (4.33)

This can be proved by projection as in (4.31).

If Σ, Σ' cut in O, O′, the centre of Γ is O′. ... (4.4)

To prove this, let O′A cut Σ' in P. Then by (2.1) PB′, PC′ are respectively parallel to AB, AC. The polars of P, B′, C′ with respect to Γ form a triangle in perspective with the tiangles PB′C′. But PB′ and AB (the polar of C′) are parallel, and likewise, PC′ and AC (the polar of B′). Hence B′C′ must be parallel to the polar of P. So O′AP is a diameter of Γ since the polars of A and P are parallel. Hence the result (3.5) row takes the form:—

If Σ, Σ' be two *s. s, p.* conics through O and if OAA′, OBB′, OCC′ be *any* three chords, the centre of the conic for which the triangles ABC, A′B′C′ are reciprocal is always at the *same* point. ... (4.41)

*The necessary and sufficient condition that Γ shall be a parabola is that $A''B''C''$ should be collinear.** ... (4.5)

The condition is necessary. For let Γ be a parabola. Then O′ is at infinity by (4.4). So Σ, Σ' are either both parabolas or both hyperbolas. If both be parabolas, then the polar axis which is their common chord must be parallel to their axes. But then the parabola Σ'' touches at O a line parallel to its axis. That is, Σ'' degenerates into the polar axis and another line parallel to it. The other line must contain A″, B″, C″.

If both Σ, Σ' be hyperbolas, then the polar axis which is their common chord must be parallel to an asymptote. But then the

* The necessity of this condition is stated in Qn. 1651 (*a*) (Vide *The Math. Student* Vol. I, No. 2).

hyperbola Σ'' touches at O a line parallel to one of its asymptotes. That is, Σ'' degenerates into the polar axis and another line which, as before, should contain A'', B'', C''.

To prove that the condition is sufficient, we observe that since A'', B'', C'' are collinear, Σ'' must degenerate into the line $A''B''C''$ and the polar axis, by (3.4). If $A''B''C''$ and the polar axis are parallel then Σ, Σ' are both parabolas and OO' their common chord being parallel to their axes, O' is at infinity and Γ is a parabola. If $A''B''C''$ and the polar axis be not parallel, then Σ, Σ' are both hyperbolas and OO' their common chord *i.e*, the polar axis) being parallel to an asymptote, O' is again at infinity, and as before Γ is a parabola.

It may be noted that the result applies only to the case where O is at a finite distance.

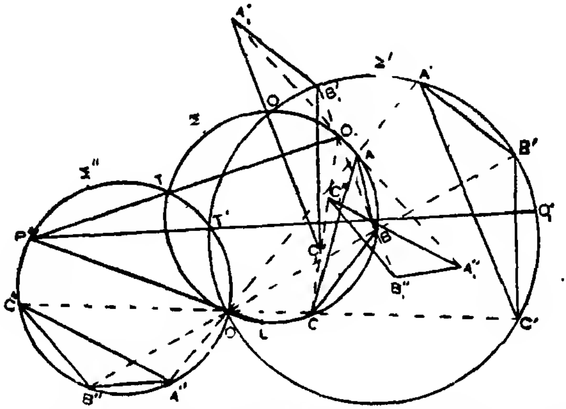
5. Displacement of triangles in perspective.

If the triangle $A'B'C'$ be given a translatory motion represented by OP where P is any point on Σ'', its perspectivity with the triangle ABC is maintained; the new centre of perspective O_1 is also on Σ; the triangle corresponding to the triangle $A''B''C''$ and the s. s. p. conics associated with the three triangles are all entirely unaltered in shape and size, being in fact the original ones merely displaced. ... (5.1)

To prove this, let Σ, Σ''; Σ', Σ'' respectively cut again in T, T'. (Refer to Fig. 1 next page),

Then, taking *any* point P on Σ'', let PT, PT' cut Σ, Σ' respectively in O_1, O_1'. Then from (2.1), PA'', PB'', PC'' are respectively parallel to O_1A, O_1B, O_1C, as well as to $O_1'A'$, $O_1'B'$, O_1C'. So if parallels to OP through A', B', C' cut O_1A, O_1B, O_1C in A_1', B_1', C_1', then each of the pairs of triangles $A'AA'_1$, $OA''P$ is congruent, since $A'A = OA''$, etc. So $A'A'_1 = B'B' = C'C'_1 = OP$. Also the triangles $A'B'O'_1$, $A_1'B_1'O_1$ are congruent and similarly placed, since $A'B' = A_1'B_1'$, etc., and $O_1A'_1$ is parallel to $O_1'A$, etc.; so $O'_1O_1 = OP$. In fact the tetrad $O_1A_1'B_1'C_1'$ may be considered as got by displacing the tetrad $O_1'A'B'C'$ by the vector $O_1'O_1$ or OP. Also since $A_1'A = PA''$, etc., from the pairs of congruent triangles $A_1'AA'$, $PA''O$, etc., if A_1'', B_1'', C_1'' be taken on O_1A, O_1B, O_1C so that $A_1'A = O_1A''$, etc., then the tetrad $O_1A_1''B_1''C_1''$ is got by displacing

the tetrad $PA''B''C''$ along the vector PO_1. So, if Σ' and Σ'' be respectively displaced along the vectors OP and PO_1, they should



[The figure is drawn for the case when Σ and Σ' are circles but the reasoning is quite general]

FIG. 1.

circumscribe $O_1A_1'B_1'C_1'$ and $O_1A_1''B_1''C_1''$ respectively. Since Σ is *s. s. p.* to Σ', Σ'' in their new positions also, it follows, by (3.3), that these three are the *s. s p.* conics in the new position. Hence the result.

The triangle ABC and Σ are fixed; the triangle $A'B'C'$ and Σ' receive a translation along OP; and the triangle $A''B''C''$ and Σ'' are displaced by PO_1. In fact the *s.s.p.* conics are simply (rigidly) *carried along* by the respective triangles Also, if PO cut Σ again in L,

The centre of Γ *in the new position is L.* ... (5.11)

For, by (4.4) and (3.4), the centre required is the point where the polar axis (in the new position) cuts Σ in addition to O_1. But, by (3.4'), this polar axis is the tangent at O_1 to Σ'' (in the new position). Also the point in the old position of Σ'' displaced to O_1 is P. Hence the required centre is the other end of the chord (of Σ) through O_1 parallel to the tangent at P to Σ''. Since, by (2.1), PO contains this point, the result follows.

We can also show that :—

The envelope of the polar axis is a fixed conic s. s. p. to and concentric with Σ. ... (5.12)

Project Σ, Σ', Σ'' into circles. Let Fig. 1 be taken as that for the projections. Let H, H′, H″ be the centres of Σ, Σ', Σ''.

Since OO′ touches Σ'' at O, O_1O' and PO or OL are parallel by (2.1). So O_1L = OO′. Hence the polar axis i nthe new position (*i.e.*, O_1L) touches the circle with centre H and radius the distance of H from OO′. The diameter of this circle is accordingly equal to the chord intercepted by Σ on H″O. Hence the result. Since HH′ bisects OO′ and is parallel to H″O, it also follows that the new conic is got by reducing each semi-diameter of Σ in the ratio of the chord intercepted by Σ on the diameter of Σ'' through O, the diameter, of Σ on the line of centres of Σ, Σ'.

Since the envelope reduces to a point when the chord reduces to a point, and since when H″O touches Σ, HO touches Σ'', it follows that :—

The polar axes are concurrent if the diameter of Σ *(or* Σ''*) through a common point touches* Σ'' *(or* Σ*).* ... (5.13)

If OABC, O′A′B′C′ be each concyclic, the centre of Γ *in each displaced position is at the same distance from the corresponding centre of perspective.* ... (5.14)

It also follows that if in one position of the triangle A′B′C′, Γ is a parabola, then, in *any* other position also Γ is *still* a parabola with a *parallel* axis provided the centre of perspective in each case is not at infinity. (5.2).

For, from (4.5), A″B″C″ are collinear in the first position and from (5.1), they are still collinear in the second position. Consideration of Σ'' (which in this case breaks up into the line A″B″C″ and the polar axis) shows by (5,1) that the polar axis and the line A″B″C″ maintain their directions. But the polar axis determines the direction of the axis of Γ. Hence the result.

6. Discussion of Question 1648 of Mr. V. Ramaswamy Aiyar.

Question 1648. (*V. Ramaswamy Aiyar.*) :—Two triangles ABC, A′B′C′ are in perspective. If A′B′C′ be moved rigidly and without rotation, so as to be constantly in perspective with ABC, show that the locus of any point P of its plane (of its *body*, if you please) is, in general, a conic.

In what case or cases will the locus of P be a line, apart from the obvious case where the centre of perspective O is at infinity when P can move straight towards O?

(5.1) indicates that any point P on Σ'' (the P in the question is taken at O) satisfies the conditions of the problem. It only remains to show that *any* point P satisfying the condition lies on Σ''. Now since OP and OA$''$ are respectively parallel and equal to $A'A'_1$ and $A'A$ it follows that AA'_1 is parallel to A$''$P and likewise BB'_1 to B$''$P and CC'_1 to C$''$P. Thus the parallels to PA$''$, PB$''$, PC$''$ through A, B, C respectively are concurrent. [Refer to Fig. 1.]

Take the conic S$''$ through O, A$''$, B$'$, C$''$, P; and through O, A, B draw S *s.s.p.* to S$''$. Draw the chord AO_2 of S parallel to PA$''$. Then, by (2.1), O_2B is parallel to PB$''$. So O_2 must be the point of concurrence of the parallels through A, B, C. Since the join of O_2 to the intersection of S and OC$''$ must be parallel to PC$''$ or O_2C, it follows that S passes through C also. Hence by (3.2) $S \equiv \Sigma$, $S'' \equiv \Sigma''$. Hence follows the first part.

We may add that the locus of P is the conic *circumscribing A$''$B$''$C$''$ and touching the polar axis.* By (24) its semi-diameter through O is equal to the distance between the centres of Σ, Σ'. ... (6.1)

The form of Σ, Σ' determines the form of Σ''. For example:—

If OABC, OA$'$B$'$C$'$ be each a cyclic quadrilateral, the locus of P is a circle. (6.11)

If only a single pair of corresponding sides, say AB, A$'$B$'$ be parallel, Σ'' breaks up into A$''$B$''$ and OC. (6.12)

If A$''$, B$''$, C$''$ be collinear, Σ'' breaks up into the line A$''$B$''$C$''$ and the polar axis. (6.13)

If O is the ortho-centre of the triangle ABC (or A$'$B$'$C$'$), Σ'' is a rectangular hyperbola breaking up into a pair of perpendicular straight lines if in addition A$''$, B$''$, C$''$ are collinear. (6.14)

(6.12), 6.13) provide the solution for the second part. In each case the locus of P is a *pair* of lines. Since $A_1'A$, $B_1'B$, $C_1'C$ are respectively parallel to PA$''$, PB$''$, PC$''$, it follows that:—

When A$''$, B$''$, C$''$ are collinear, there are two types of displacements. For *any* displacement OP where P is on the line A$''$B$''$C$''$, the new centre of perspective is at infinity, A_1', B_1', C_1' respectively lying on the parallels to A$''$B$''$C$''$ through ABC; for all *other* dis-

placements, *i.e.*, those parallel to the polar axis, the new centre is *not* at infinity. ... (6.15)

This result may be useful to explain the positions in (5.2)

The case when O is at infinity.

We shall prove that in this case displacements *other* than those parallel to AA′ are possible.

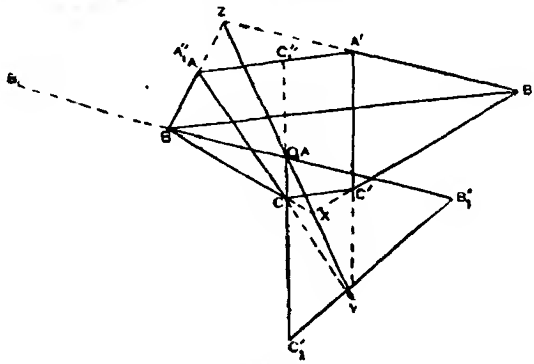


FIG. 2.

Draw BO_1, CO_1 parallel to B′A′, C′A′ respectively (Fig. 2.)* Displace the triangle A′B′C′ by $A'O_1$. Then A_1' is at O_1, and B_1', C_1' are respectively on BO_1 and CO_1. Hence O_1 is the new centre of perspective.

It is easy to see that A_1'' is at A and that O_1B, O_1C cut AA′ in B_1'', C_1''. The centre of perspective being at a vertex, the family (F_1') should be taken as the one circumscribing the triangle $A'_1B'_1C'_1$ and *touching* A'_1A. The locus in this new position is accordingly the line $A''_1B''_1C''_1$ and the polar axis. But since every possible position of A′B′C′ is a possible position of $A'_1B'_1C'_1$ and conversely, it follows that the locus in the original position is also the same for displacements to be given to the point A′. We can show that

The polar axis in the second position is the axis of perspective in the first. (6.21)

For O_1 is the centre of Γ (in the first position). Because the triangle O_1BC and its reciprocal triangle are in perspective, and O_1B and the polar of C (*i.e.*, A′B′) are parallel and O_1C and the polar of B (*i.e.*, A′C′) are parallel, it follows that the polar of O_1 is parallel to

* The triangle BO_1C is fixed for all displacements of the triangle A′B′C′ parallel to AA′. In the figure the triangle A′B′C′ is drawn for the *particular* position for which the axis of perspective *contains* O_1.

BC. So $A'O_1$ is a diameter. Since XYZ itself is also a diameter, O_1 is the centre. Again, denoting by $\Omega_{A'B'}$, $\Omega_{A'C'}$ the points at infinity on A′B′ and A′C′, the triangles $BZ\Omega_{A'B'}$ and $C\Omega_{A'C'}C'$ are reciprocal for Γ (the line $C\Omega_{A'C'}$ is only the diameter CO_1). Hence BC and the parallels through Z and C′ to A′C′ and A′B′ respectively, are concurrent, say at C_1'. Since $C'C_1' = A'Z$, the displacement of the triangle A′B′C′ by A′Z maintains perspectivity. The new centre of perspective is the point B, and this not being at infinity, it follows that Z lies on the polar line in the second position (*i.e.*, where O_1 is the centre of perspective). Hence the result (6.21). Accordingly we see that :—

When O is at infinity, the locus is a pair of lines, *i.e.*, A′ can receive displacements parallel to AA′ as well as those represented by vectors from A′ to the axis of perspective XYZ. (6.22)

Incidentally from (6.21), it follows that :—

If two triangles in perspective have their centre of perspective at a finite point and if when one of the triangles be displaced, perspectivity is maintained but the new centre of perspective is at infinity, then the polar axis in the first position is parallel to the axis of perspective in the second position. (6.23)

Geometric determination of the possible displacement of the triangle A′B′C′ in any given direction.

When Σ'' is a circle or a pair of lines, the construction is obvious. The following covers the general case.

Let O′A′, O′A cut Σ, Σ' respectively in P, P′. Let OP cut Σ' in P′. Then, by (2.1), PB, PC are respectively parallel to A′B′, A′C′ and (P′) B′, (P′) C′ are respectively parallel to AB, AC. Hence P, (P′) can be constructed.* Also (P′) P′ is parallel to AP and hence P′ and therefore P″ where P′P = OP″ can be got. Now O, A″, B″, C″, P″ being five points on Σ'', the point X″ on it lying on any specified line OX can be got from the equality of the cross-ratios, O (X″ A″ B″ C″) and P″ (X″ A″ B″ C″) by the usual methods. OX″ is the required displacement.

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* This method of constructing P, Q, R and (P′), (Q′), (R′) defines Σ Σ' irrespective of the position of O. Also the locus, say of A′ (for displacements) is Σ'' displaced by OA′. So even when O is at infinity we get Σ, Σ' and the locus of A′ should be *s. s. p.* to them. But since the ends of the vectors from A′ equal to A′A, B′B, C′C all lie on A′A, the locus of A′ should degenerate into a *pair* of lines parallel to the asymptotes of Σ, Σ' confirming the preceding discussions of the case when O is at infinity.

THE ARCHIMEDIAN SOLIDS.

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1. An Archimedian solid is a convex polyhedron, whose faces are regular polygons, not all similar, and whose solid angles are all congruent

The present paper contains, what is believed to be a new proof of the theorem, that there cannot be more than fifteen different types of these solids and also a simple method for the construction of card-board models.

As the figure is convex, the plane-angle sum at a vertex is less than $360°$; and hence

(1) The polygons at a vertex cannot belong to more than three different species, for the least angle-sum for four species is $60° + 90° + 108° + 120° = 378°$, and

(2) the number of polygons at a vertex must not exceed five, for if it be six, the least value of the angle-sum would be $5 \times 60° + 90° = 390°$.

2. Theorem A. If there are three polygons at a corner and the number of sides in one of these is odd, then the two remaining polygons must be similar and must have an even number of sides.

Let P_α, P_β and P_γ denote regular polygons of angles, α, β and γ. Let A_1A_2 and A_2A_3 be two consecutive edges of P_α and let the other polygon standing on A_1A_2 be P_γ. Then P_β must be the polygon on A_2A_3, P_γ the polygon on A_3A_4 and so on. But this is possible only if P_α has an even number of sides. If this number be odd, then we must have $\beta = \gamma$.

Theorem B. If there are three polygons at a corner and two of these are similar, then each of these two polygons must have an even number of sides, (for otherwise there will be a

contradiction to Theorem A), and if all the polygons are dissimilar, then each of them must have an even number of sides.

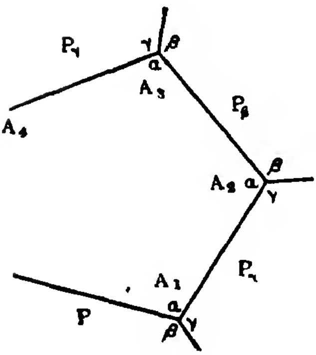

FIG. 1.

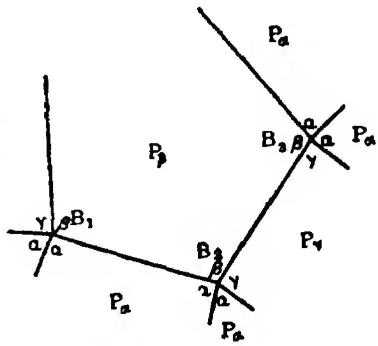

FIG. 2.

THEOREM C. If there are four polygons at a corner, then two at least must be similar (otherwise the angle-sum will exceed 360°) and these cannot be adjacent if one of the remaining polygons has an odd number of sides.

Let the polygons be P_α, P_α, P_β, and P_γ and let P_β be the polygon with an odd number of sides. If possible let the order of the angles at B_1 be $\beta\gamma\alpha\alpha$; then the order at B_2 must be $\beta\alpha\alpha\gamma$. Thus the polygon on the other side of B_1B_2 is P_α; the polygon on the other side of B_2B_3 is P_γ, and so on. But this is impossible when P_β has an odd number of sides.

THEOREM D. If there are four polygons at a vertex and only two of these are triangles, then the two remaining polygons must be similar and non-adjacent.

For if P_α is a triangle, the two equal angles α (= 60°) may be (1) adjacent or (2) opposite. But going round the corners in the order $A_1A_2A_3$ it will be found that in either case, the polygon on the other side of A_3A_1 will not be regular unless two of the angles α, β, γ are

equal. But if there are only two triangles, α cannot be equal to β or γ. Therefore $\beta = \gamma$, *i.e.*, $P_\beta = P_\gamma$.

Theorems A, B, C, D together with Euclid's theorem that the sum of the plane angles at a vertex is less than 360° (which will be referred to as Theorem E) enable us to find out the possible types of the Archimedian solids. We consider the polygons at a corner and classify the figures as follows :

I. *Solids with four triangles at a vertex.*

There can be only one more polygon which may be a square or a pentagon (E). The two corresponding figures may be conveniently denoted by the symbols $3_4 4_1$ (the Snub Cube) and $3_4 5_1$ (the Snub Dodecahedron) there being four triangles and one square at a vertex of the first figure and four triangles and one pentagon at a vertex of the second.

II. *Solids with three triangles at a vertex.*

There can be only one more polygon and it may have any number of sides (E). This figure is called the Semiregular Prismoid, its symbol being $3_3 n_1$.

III. *Solids with two triangles at a vertex.*

The number of remaining polygons cannot be one (B). Therefore there must be two more polygons and these must be both squares or both pentagons (B). The two corresponding figures are the Cuboctahedron $3_2 4_2$ and the Icosidodecahedron $3_2 5_2$.

IV. *Solids with one triangle at a vertex.*

(*a*) There may be two more polygons or (*b*) there may be three.

(*a*) If there are only two, they must be similar polygons with an even number of sides (A). Therefore they must be both squares, both hexagons, both octagons or both decagons (E). Thus we get the solutions $3_1 4_2$. (The triangular prism included in the general class of prisms V); $3_1 6_2$ (the truncated Tetrahedron); $3_1 8_2$ (the Truncated Cube) and $3_1 10_2$ (the Truncated Dodecahedron),

(*b*) If there are three more polygons, they cannot be all dissimilar (E). Therefore they may be (1) all similar, in which case each must be a square (E) or (2) only two of them may be similar (C) in which case we must have two squares and one pentagon (E). The two corresponding figures are $3_1 4_3$ (the Small Rhombicuboctahedron) and $3_1 4_2 5_1$ (the Small Rhombicosidodecahedron).

V. *When there are no triangles.*

The greatest number of squares at a vertex is two; and in this case there will be only one more polygon (E) which may have any number of sides. This figure is the Semiregular Prism $4_2 n_1$.

If there is only one square (but no triangle), there can be only two more polygons (E). If these are similar, they must be both hexagons (B, E) and we get the figure called the truncated Octahedron $4_1 6_2$. If the two polygons are dissimilar, they must have an even number of sides (B); and there may be one hexagon and one octagon, or one hexagon and one decagon, other combinations being impossible (E). The two corresponding figures are the Great Rhombicuboctahedron $4_1 6_1 8_1$ and the Great Rhombicosidodecahedron $4_1 6_1 10_1$.

VI. *When there are no triangles and no squares.*

The greatest number of pentagons at a corner is two (E), but then there will be only one polygon more, and this combination is impossible (B). Hence we must have only one pentagon and two hexagons (B, E). This is the Truncated Icosahedron $5_1 6_2$.

VII. *When there are no triangles, squares or pentagons.*

The least value of the angle-sum will be greater than 360° and no figure is possible in this case.

This proves that there are only fifteen possible types of Archimedian solids,* *viz.*, (1) $3_4 4_1$, (2) $3_4 5_1$, (3) $3_3 n_1$, (4) $3_2 4_2$, (5) $3_2 5_2$, (6) $3_1 6_2$, (7) $3_1 8_2$, (8) $3_1 10_2$, (9) $3_1 4_3$, (10) $3_1 4_2 5_1$, (11) $4_2 n_1$, (12) $4_1 6_2$, (13) $4_1 6_1 8_1$, (14) $4_1 6_1 10_1$, (15) $5_1 6_2$.

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* It may be shown that if F, V, E be the numbers of faces, vertices, and edges of a polyhedron at each vertex of which meet a polygons with α sides, b polygons with β sides and c polygons with γ sides, then

$V = 2/(1 + \Sigma\, a/\alpha - \tfrac{1}{2}\Sigma\, a)$; $E = \tfrac{1}{2} V(a+b+c)$; $F = 2 + E - V = 2 + \tfrac{1}{2} V(a+b+c-2)$

A table of values is given in the article on Geometrical Solids, *Encyclopaedia Britannica*, (Fourteenth Edition).

Each of these figures has got a circumscribed sphere and the radius of this sphere can be calculated after determining the dihedral angles by the methods of spherical trigonometry.

It is easy to construct card-board models of these figures by the method described below. The card-board must be flexible and not very thick. The ordinary brown card-board is not suitable, but the paper of visiting cards may be used.

Let P be the polygon which occurs the least number of times (p say) in the whole figure. The arrangement of the other dissimilar polygons surrounding P may be found by considering the form of the solid angles at the corners of P. On the surface of the solid, P and the surrounding polygons will form a cap-like figure. Let this be developed into a plane figure, and drawn on a piece of card-board. Join the other extremities of each pair of edges which meet at a corner of P (but do not belong to P) and cut the paper along the boundary of the figure so formed. It will be found that there are certain triangular areas between two neighbouring polygons, which do not belong to the solid; but these are not to be cut off. Now, if the central polygon P has n sides, the surrounding polygons may be divided into n groups, one group belonging to each side of P.

These groups are to be separated from one another (but not from the central polygon P) by cutting the paper along the proper edges, and creases are to be formed by folding the paper along all the lines in the figure. Then the triangular part outside each group, is to be pasted over the adjacent polygon of the next group (on the concave side of the figure). When this is done for all the groups, a cap will be formed, and p such caps fitted together and pasted one over the other, will form the required polyhedron. In pasting one part of the figure over another, it is necessary to fix the two parts in the proper position until the paste is dry; and for this purpose clips will have to be used. Again, before folding the paper along any line, it should be ruled along that line with the sharp point of the divider.

By way of illustration, let the required polyhedron be $3_1 4_2 5_1$ (the Small Rhombicosidodecahedron).

Fig. 3 shows the plane development of a cap. The superfluous triangular areas are AP_1T_8, BQ_1P_8, etc. The paper is to be cut

along the lines AP_1, BQ_1, CR_1, DS_1 and ET_1 and to be folded along all the other lines in the figure. The triangle AP_1T_3 is then to be pasted over the square AP_1P_2B; the triangle BQ_1P_3 over the square BQ_1Q_2C and so on. Twelve such caps can be fitted together to form the required polyhedron.

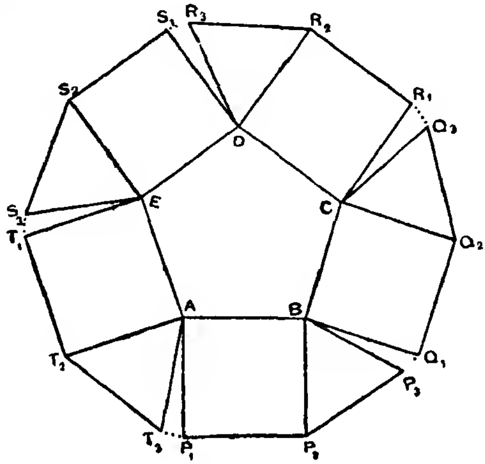

FIG. 3.

This method is applicable to any convex polyhedron, but can be conveniently modified when some of the surrounding polygons are very large and also when the number of vertices is smaller than the number of faces. In this latter case, we form pyramidal caps having the shape of the solid angle of the figure, and these caps are fitted together. With some modifications the method can also be used to construct card-board models of the concave polyhedra obtained by stellating the convex regular solids.

Other materials which may be used to construct models are celluloid wires, sticks, strings, clay, wood, etc. But celluloid is costly, wire and strings do not give coloured models, and it is difficult to work with wood or clay.

. 'Life and cosmos are periodic continued fractions, and hence both are irrational.'

'*Numerology*,' by E. T. BELL,
(Per Mr. A. A. KRISHNASWAMI AYYANGAR).

AN EXTENSION OF HEILBRONN'S CLASS-NUMBER THEOREM.

BY S CHOWLA.

Let $h(d)$ denote the number of primitive classes of binary quadratic forms of negative discriminant d. Heilbronn has recently shown that

THEOREM A.

$$h(d) \to \infty \quad as \; -d \to \infty.$$

This result was conjectured by Gauss.

I have succeeded in proving the following extension (also conjectured by Gauss) of Theorem A.

THEOREM B. *If* t *denotes the number of different prime factors contained in* d, *then*

$$\frac{h(d)}{2^t} \to \infty, \quad as \quad -d \to \infty.$$

We use the method used by Heilbronn to prove Theorem A. The important new weapon is the following theorem of Pepin:

THEOREM. *Let* d *be quadratfrei and*

$$a/(x^2 + d), \qquad (a, 2d) = 1.$$

Then

$$a^{2p} = x^2 + dy^2 \qquad (y \neq 0$$

where p *is the number of classes in the principal genus of reduced primitive non-equivalent binary quadratic forms of negative determinant* —d.

This theorem is quoted by Dickson, *History of the theory of numbers*, Volume 3, page 58. Details of the proof of Theorem B will appear elsewhere.

[Received July 18, 1934.]

NOTES AND DISCUSSIONS.

The Editor welcomes for publication under this heading, brief discussions of interesting problems, critical comments, and suggestions likely to be helpful in the class-room.

On the Radius of a circle and the condition of orthogonality of two given circles.

In this note I obtain expressions for the radius of a circle whose equation is given in areal co-ordinates, from the theorem that the numerical value of the power of the centre with respect to the circle is equal to the square of the radius.

Let $\phi(x, y, z) \equiv \Sigma\, ux^2 + 2\,\Sigma\, fyz = 0,$

be the equation of a circle in areal co-ordinates, so that

$$\frac{v+w-2f}{a^2} = \frac{w+u-2g}{b^2} = \frac{u+v-2h}{c^2} = k \quad \text{(say)}; \quad \ldots \quad (1)$$

then, the power of the point $P(x_1, y_1, z_1)$ with respect to the circle is

$$\phi(x_1, y_1, z_1)\,/\,k \qquad \ldots \quad (2)$$

If the point $P(x_1, y_1, z_1)$ is the centre, then

$$ux_1 + hy_1 + gz_1 - \lambda = 0; \quad hx_1 + vy_1 + fz_1 - \lambda = 0;$$
$$gx_1 + fy_1 + wz_1 - \lambda = 0; \quad x_1 + y_1 + z_1 - \lambda = 0$$

$$\text{Hence} \begin{vmatrix} u & h & g & \lambda \\ h & v & f & \lambda \\ g & f & w & \lambda \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0, \; i.e. \begin{vmatrix} u & h & g & 1 \\ h & v & f & 1 \\ g & f & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} \lambda + \Delta = 0 \ldots \quad (3)$$

where Δ is the discriminant of ϕ.

The determinant in the last equation equals

$$\frac{1}{2}\begin{vmatrix} 2u & u+v-kc^2 & w+u-kb^2 & 1 \\ u+v-kc^2 & 2v & v+w-ka^2 & 1 \\ w+u-kb^2 & v+w-ka^2 & 2w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = -4k^2 S^2$$

where S is the area of the triangle of reference. Therefore $\lambda = \Delta/4S^2k^3$.

Also $$\phi\,(x_1, y_1, z_1) = \tfrac{1}{2}\,\Sigma\, x_1 \frac{\partial\phi}{\partial x_1} = \lambda.$$

Hence, if ρ be the radius of the circle,

$$-\rho^2 = \frac{\lambda}{k} = \frac{\Delta}{4S^2k^3} \qquad \ldots \quad (3)$$

On substituting for Δ and using (1), this result may also be written in the form

$$\rho^2 = \frac{\Sigma\, u^2a^2 - 2\,\Sigma\, bc\cos A\ vw - 2\,abc\,k\,\Sigma\, ua\cos A + k^2a^2b^2c^2}{16S^2k^2} \qquad \ldots \quad (4)$$

Condition of orthogonality of two circles.

Let $\phi_r\,(x, y, z) \equiv \Sigma\, u_rx^2 + 2\,\Sigma\, f_r\, yz \equiv 0 \qquad (r = 1, 2)$

be two circles, so that

$$\frac{v_r + w_r + 2f_r}{a^2} = \frac{w_r + u_r - 2g_r}{b^2} = \frac{u_r + v_r - 2h_r}{c^2} = k_r.$$

Suppose (x_r, y_r, z_r) is the centre and ρ_r is the radius of the circle ϕ_r; then

$$-\rho_1^2 = \frac{\phi_1\,(x_1, y_1, z_1)}{k_1}; \qquad d^2 - \rho_2^2 = \frac{\phi_2\,(x_1, y_1, z_1)}{k_2} \quad \ldots \quad (5)$$

where d is the distance between the centres of the circles. If the circles cut orthognally

$$d^2 = \rho_1^2 + \rho_2^2 \qquad \ldots \quad (6)$$

so that by (5), the centre $(x_1\, y_1\, z_1)$ of ϕ_1 lies on the radical circle

$$\phi_1/k_1 + \phi_2/k_2 = 0.$$

Similarly (x_2, y_2, z_2) lies on this circle. Consequently if the circles ϕ_1 and ϕ_2 cut orthogonally, the radical circle of the given circles passes through the centres of both the circles.* Since it also passes through the common points of the circles, the line joining the centres of ϕ_1 and

* Prof. Rao's note in the *Bulletin of the Calcutta Mathematical Society*, Vol. XXI, 1929, pp. 195-6.

ϕ_3 is a diameter of the radical circle. Thus if ρ_3 be the radius of the radical circle, the condition of orthogonality of the circles ϕ_1 and ϕ_2 can be put in the form $\rho_1^2 + \rho_2^2 = 4\rho_3^2$. Hence from (4)

$$\frac{1}{16S^2k_1^2}\left[\Sigma\, u_1^2a^2 - 2\,\Sigma\, bc\cos A\, v_1w_1 - 2\,abc\,k_1\,\Sigma\, u_1a\cos A + k_1^2a^2b^2c^2\right]$$

$$+\frac{1}{16S^2k_2^2}\left[\Sigma\, u_2^2a^2 - 2\,\Sigma\, bc\cos A\, v_2w_2 - 2\,abc\,k_2\,\Sigma\, u_2a\cos A + k_2^2a^2b^2c^2\right]$$

$$=\frac{1}{16S^2}\left[\Sigma\left(\frac{u_1}{k_1}+\frac{u_2}{k_2}\right)^2a^2 - 2\,\Sigma\, bc\cos A\left(\frac{v_1}{k_1}+\frac{v_2}{k_2}\right)\left(\frac{w_1}{k_1}+\frac{w_2}{k_2}\right) - 4abc\,\Sigma\left(\frac{u_1}{k_1}+\frac{u_2}{k_2}\right)a\cos A + 4a^2b^2c^2\right].$$

This reduces to

$$\Sigma\, a^2u_1u_2 + a^2b^2c^2\,k_1k_2 = \Sigma\, bc\cos A\,(v_1w_2 + v_2w_1) + abc\,\Sigma\,(u_1k_2 + u_2k_1)\,a\cos A.$$

The advantage of the method is that while using the usual condition of orthogonality (6), we have avoided the calculation of d and used instead the fact that d is the diameter of the radical circle.

S. D. College, Lahore. HUKAM CHAND.

Recurrence Formulae for $J_n(x)$.

In this note I suggest proofs of the well-known relations (i) to (iv) connecting Bersel functions, which are simpler than those usually given in usual text-books.

(i) $2n\,J_n = x\,(J_{n-1} + J_{n+1})$

(ii) $2\,\dfrac{d}{dx}J_n = J_{n-1} - J_{n+1}$

(iii) $x\,\dfrac{d}{dx}J_n = n\,J_n - x\,J_{n+1}$

(iv) $x\,\dfrac{d}{dx}J_n = -\,n\,J_n + x\,J_{n-1}$

To prove (i) we start with the definition of $J_n(x)$, namely,

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - x\sin\phi)\, d\phi \qquad \ldots \quad (1)$$

[See Ganesh Prasad's *Spherical Harmonics*, Article 64.]

$$\therefore \qquad J_{n+1} = \frac{1}{\pi}\int_0^{\pi} \cos(\overline{n+1}\,\phi - x\sin\phi)\, d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos(\psi + \phi)\, d\phi \qquad \ldots \quad (2)$$

$$J_{n-1} = \frac{1}{\pi}\int_0^{\pi} \cos(\overline{n-1}\,\phi - x\sin\phi)\, d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos(\psi - \phi)\, d\phi \qquad \ldots \quad (3)$$

$$J_n = \frac{1}{\pi}\int_0^{\pi} \cos\psi\, d\phi \qquad \ldots \quad (4)$$

where $\psi = n\phi - x\sin\phi$ $\qquad \ldots \quad (5)$

From (2) and (3) we get by addition

$$J_{n+1} + J_{n-1} = \frac{2}{\pi}\int_0^{\pi} \cos\psi\cos\phi\, d\phi \qquad \ldots \quad (6)$$

Again from (5), $d\psi = (n - x\cos\phi)\, d\phi$, so that we get

$$\frac{1}{\pi}\int_0^{\pi} \cos\psi\, d\psi = \frac{n}{\pi}\int_0^{\pi} \cos\psi\, d\phi - \frac{x}{\pi}\int_0^{\pi} \cos\psi\cos\phi\, d\phi.$$

But as $\int_0^{\pi} \cos\psi\, d\psi = 0$ and $\frac{1}{\pi}\int_0^{\pi} \cos\psi\, d\phi = J_n$ by (4),

it follows that

$$n J_n = \frac{x}{\pi}\int_0^{\pi} \cos\psi\cos\phi\, d\phi. \qquad \ldots \quad (7)$$

The result (i) follows from (6) and (7).

Again, on differentiating (1) with respect to x we get

$$\frac{d}{dx} J_n = \frac{1}{\pi} \int_0^{\pi} \sin(n\phi - x \sin\phi) . \sin\phi \, d\phi = \frac{1}{\pi} \int_0^{\pi} \sin\psi \sin\phi \, d \quad \ldots \quad (8)$$

But, from (3) and (2) by subtraction we get

$$J_{n-1} - J_{n+1} = \frac{2}{\pi} \int_0^{\pi} \sin\psi \sin\phi \, d\phi$$

whence, using (8), we get result (ii).

To prove the relations (iii) and (iv) we start with (ii) which we have just now proved.

$$2x \frac{d}{dx} J_n = x J_{n-1} - x J_{n+1} \quad \ldots \quad (9)$$

Again $$2n J_n = x J_{n-1} + x J_{n+1} \quad \ldots \quad (10)$$

Eliminating J_{n-1} between (9) and (10) we get the result No. (iii), while eliminating J_{n+1} between (9) and (10) we get the result No. (iv).

Bhagwan Ashram, Tundla (Agra, U.P.)

RATAN LAL GUPTA, M.Sc.

Lecture Notes.

On the identity $(\sum_1^N n)^2 = \sum_1^N n^3$.

The usual text-book proofs of this result are by the method of induction. The result may, however, be directly verified as follows :—

LEMMA 1. $(a + b + c + \ldots l + k)^2 = a^2 + b^2 + c^2 + \ldots k^2$
$+ 2a(b + c + \ldots k)$
$+ 2b(c + d + \ldots k)$
$+ \ldots \ldots \ldots \ldots$
$+ 2lk.$

LEMMA 2. $m^2 + 2m(\overline{m-1} + \overline{m-2} + \ldots 2 + 1) = m^3.$

Now

$$\begin{aligned}(1+2+3\ldots n)^2 &= (n+\overline{n-1}+\overline{n-2}+\ldots 2+1)^2\\ &= n^2+2n\,(\overline{n-1}+\overline{n-2}+\ldots 2+1)\\ &+ (n-1)^2+2\,(n-1)\,(\overline{n-2}+\overline{n-3}+\ldots 2+1)\\ &+ (n-2)^2+2\,(n-2)\,(\overline{n-3}+\overline{n-4}+\ldots 2+1)\\ &+ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\ &+ 2^2+2.2.(1)\\ &+ 1^2 \qquad\qquad \text{by Lemma 1}\\ &= n^3+(n-1)^3+(n-2)^3+\ldots 2^3+1 \text{ by Lemma 2.}\end{aligned}$$

Trichy. S. SURYANARAYANA.

Multiplication of numbers ending in 5.

It is useful to notice that the product of two numbers ending in 5, say $a5$ and $b5$ (where a and b are any rows of figures) is given by

$$a5 \times b5 = \left\{ a \times b + \left[\frac{a+b}{2}\right] \right\} c$$

where c stands for the figures 25 or 75 according as $a \sim b$ is even or odd, and $[x]$ denotes the greatest integer in x.

$$\text{Thus} \quad 1175 \times 1325 = \left[117 \times 132 + \left\{\frac{117+132}{2}\right]\right\} 25$$

$$= 1556875.$$

Vizagapatam. K. SUBBA RAO.

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27. 'A problem may be solved by merely human intelligence, but a theorem (derived from '*Theos*' God and '*res*' a thing) requires divine assistance.'

Quoted by DEAN INGE in '*Lay Thoughts of a Dean.*'
(Per Mr. A. A. KRISHNASWAMI AYYANGAR).

28. 'Our account with our environment may be represented by a vulgar fraction, the numerator being what we have and the denominator what we want We may improve this either by increasing our numerator, which is the wisdom of the West, or by diminishing our denominator, which is the wisdom of the East. Which is the right method of solving the problem?' DEAN INGE.

(Per Mr. A. A. KRISHNASWAMI AYYANGAR).

Sundaram's Sieve for Prime Numbers.

Mr. S. P. Sundaram, of Satyamangalam, is a young mathematical student (a relation of mine) who, it appears, has not been able to get through his "Intermediate." He wrote to me recently giving me an interesting theorem of his, which may be stated as follows :—

Let us form a double series whose corner term is 4. The first row is an A. P. with common difference 3. The first column is the same as the first row. The second row is an A. P. with common difference 5; the third, with common difference 7; the fourth, with 9; and so on. The double series will thus be

4	7	10	13	16	19	...
7	12	17	22	27	32	...
10	17	24	31	38	45	...
13	22	31	40	49	58	...
16	27	38	49	60	71	...
19	32	45	58	71	84	...
...	...	...	...	...	...	...

Then it will be seen that the kth column series is the same as the kth row series, for all k

Mr. Sundaram's Theorem is as follows :—*If* N *be any number of the natural series 1, 2, 3, 4, 5, ..., which occurs in the double series, then* $2N + 1$ *is a composite number. But if* N *does not occur in the double series, then* $2N + 1$ *is a prime number; and all the primes except 2 can be got in this manner.*

The Theorem strikes me as both interesting and beautiful; but, most of our readers, I think, will be able to prove it for themselves, if they try

V. RAMASWAMI AIYAR.

SOLUTIONS TO QUESTIONS.

Question 1571.

(V. Ramaswami Aiyar, M.A.):—Let M be the foot of the perpendicular drawn from a focus S on the tangent at a fixed point P of a conic. If PP', SS' be perpendiculars drawn on a variable tangent of the conic, show that $PP'.SS'$ varies as MS'^2.

Solution by A. Ranganatha Rao.

Draw PQ perpendicular to SS'. Then the fixed circle Σ described on PS as diameter passes through M and Q. From a known property of conics, CM is parallel to HP, C being the centre and H the other focus of the conic. Therefore, CM bisects PS. Thus M, a point common to the circle Σ and the auxiliary circle of the conic, lies on their line of centres. Hence the two circles touch at M. If $S'M$ cut the circle Σ again in R, then $MR : MS'$ = radius of the circle Σ : radius of the auxiliary circle = constant. Therefore $RS : MS'$ is also constant. Now,

$$\frac{PP'.SS'}{MS'^2} = \frac{QS'.SS'}{MS'^2} = \frac{RS'.MS'}{MS'^2} = \frac{RS'}{MS'} = \text{constant}.$$

Hence, $PP'.SS'$ varies as MS'^2.

Question 1574

(V. Ramaswami Aiyar, M.A.):—In a given triangle ABC, AX is drawn perpendicular to BC and two points P, Q are taken thereon such that P, Q, B, C are mutually ortho-centric. Show that the pedal circles of P, Q with respect to ABC touch one another at X. Also if T be the point in BC whose pedal circle with respect to ABC touches those of P and Q at X, then TP, TQ envelope a hyperbola of which XA, XA' are asymptotes and AA' is a tangent where AA' is the diameter through A of the circum-circle of ABC.

Solution by A. Ranganatha Rao.

Let P', Q' be the isogonal conjugates of P, Q with respect to ABC. P', Q' obviously lie on AA'. Let PQ', $P'Q$ meet in T_0 and let Ω be the point at infinity on AX.

Since $XP.XQ = BX.XC =$ constant, P, Q form a pair belonging to an involution range on XA. Now, to every point P on AX corresponds uniquely a point Q, and *vice versa*; and to a point Q on AX corresponds uniquely a point Q', on AA', and *vice versa*. Hence the envelope of PQ' (and obviously also of QP') is a conic Σ touching AX and AA'. The point of contact of Σ with AX must correspond to the point A on AA'. Now, if Q' is at A, Q is at X and P is at Ω which is, therefore, the point of contact required. Hence XA is an asymptote of Σ. Again, the point of contact of Σ with AA' must correspond to the point A on AX Now, if P is at A, Q is at the ortho-centre of ABC, and Q' is at the circum-centre O of ABC. Thus Σ touches AA' as its mid. point. If P is at X, Q is at Ω and Q' is at A'. Therefore XA' is a tangent to Σ, the point of contact being Ω', say. Since Σ is an in-conic of the triangle AXA' the points of contact with the sides being Ω, Ω', O, the lines $A\Omega'$, XO, $A'\Omega$ must be concurrent. Hence Ω' is the point at infinity on XA', *i.e.*, XA' is the other asymptote of Σ and therefore X is the centre of Σ.

Now, to solve the question, consider one position of P on AX. It is known that isogonal conjugates with respect to ABC are pairs of conjugate points with respect to the pencil of rectangular hyperbolas passing through the in- and ex-centres of ABC. Since PP', QQ' are isogonal conjugates, it follows by Hesse's theorem that A and T_0 are isogonal conjugates. Hence T_0 lies on BC. Again, consider the system of conics inscribed in the complete quadrilateral formed by the lines AX, AA', PQ', QP'. The centres of these conics lie on a line q, which passes through X, since Σ belongs to the system, and through the mid. points of PP', QQ', AT_0 since these are the point-pairs belonging to the system. The pedal circle of P with respect to ABC is the auxiliary circle of the in-conic of ABC, whose foci are P, P'. The centre of this pedal circle is, therefore, the mid. point of PP'. Thus the centres of the pedal circles of P, Q, T_0 lie on a line q passing through X which lies on the three pedal circles. Hence the three pedal circles touch at X and T_0 is no other than the point T. The envelope of TP, TQ is, as we have already seen, the conic Σ possessing the properties mentioned in the question.

Question 1593.

(B. B. BAGI) :—O is any point in the plane of a triangle ABC. BC, CA, AB, OA, OB, OC are of lengths a, b, c, a', b', c' and they make angles α, β, γ and α', β', γ' with a straight line respectively. Prove that there exists a triangle whose sides are proportional to aa', bb', cc' and whose exterior angles are λ, μ, ν where λ, μ and ν are given by

$$\left.\begin{aligned} \lambda &\equiv \gamma + \gamma' - \beta - \beta', \\ \mu &\equiv \alpha + \alpha' - \gamma - \gamma', \\ \nu &\equiv \beta + \beta' - \alpha - \alpha', \end{aligned}\right\} \text{mod. } (2\pi).$$

Give a geometrical construction for the triangle.

Solution by S. Ramaswami Iyer.

Construction. At A and B make angles BAE and ABE equal to $\angle$CAO and $\angle$ACO respectively to meet at E. Then the triangle EBO satisfies the condition of the question. For by construction, the triangles ABE, ACO are similar. Hence AB/AC = BE/CO, *i.e.*, BE = cc'/b.

Also AB/AE = AC/AO and $\angle$BAC = $\angle$EAO.

$\therefore$ the triangles A EO and ABC are similar. Hence AO/AC = EO/BC, *i.e.*, EO = aa'/b.

Hence BE : EO : BO = $cc'/b : aa'/b : b' = cc' : aa' : bb'$.

$$\begin{aligned} \angle EBO &= \angle EBA + \angle ABC - \angle OBC \\ &= \angle OCA + \angle ABC - \angle OBC \\ &= \beta - (\pi + \gamma') + \alpha - (\pi + \gamma) - (\alpha - \overline{\pi + \beta}) \\ &= (\beta + \beta') - (\gamma + \gamma') - \pi. \end{aligned}$$

Hence the exterior a ngle is

$$\pi - (\beta + \beta') + (\gamma + \nu') + \pi$$
$$= (\gamma + \gamma') - (\beta + \beta') \text{ mod. } 2\pi.$$

Similarly for the other exterior angles.

Question 1609.

(R. RAMANUJAN) :—Construct a triangle (geometrically) given the base, one of the base angles and the internal or external bisector of the vertical angle.

Solution by A. V. K. Krishna Menon.

With AB as base, describe the arc containing the angle equal to the vertical angle.

Let P be the middle point of the arc subtending the vertical angle.

Join PB and draw BQ at right angles to PB making BQ = half the given length of the bisector of the vertical angle.

With Q as centre and radius QB describe a circle. Let PQ produced cut this circle at R and PQ at S. With P as centre and radius PR describe a circle cutting the arc first drawn at C

Then ACB will be the required triangle.

Join PC cutting AB at T.

In the triangles BPT and PBC, $\angle P$ is common, and $P\hat{B}T = P\hat{C}A$
$= P\hat{C}B$.

$\therefore$ the triangels are similar and $\dfrac{CP}{BP} = \dfrac{BP}{PT}$.

$\therefore$ $BP^2 = CP.PT$. But $BP^2 = PS.PR$.

$\therefore$ $PC.PT = PS.PR$.

But $PC = PR$ [construction].

$\therefore$ $PS = PT$.

$\therefore$ $CT = SR =$ the given length of the bisector of the vertical angle.

When the external bisector is given, the same method applies. Only we have to take instead of P, its diametrically opposite point, and S instead of R.

Question 1610.

(S. MUKHOPADHYAYA) :—ABC is any triangle. BX and CX are, respectively, the internal (external) trisectors of the angles B and C nearest to BC; CY and AY are, respectively, the internal (external) trisectors of the angles C and A, nearest to CA; AZ and BZ are, respectively, the internal (external) trisectors of the angles A and B nearest to AB. Prove that the triangle XYZ is equilateral by elementary geometry.

Solution by the Proposer.

Case (i) *Internal Trisectors.*

Take any equilateral triangle XYZ. Construct triangles YUZ ZVX, XWY, externally, on YZ, ZX, XY, respectively, such that

$$\angle YUZ = \frac{A}{3}, \quad \angle UZY = \frac{B}{3} + 60^\circ, \quad \angle ZYU = \frac{C}{3} + 60^\circ;$$

$$\angle ZVX = \frac{B}{3}, \quad \angle VXZ = \frac{C}{3} + 60^\circ, \quad \angle XZV = \frac{A}{3} + 60^\circ;$$

$$\angle XWY = \frac{C}{3}, \quad \angle WYX = \frac{A}{3} + 60^\circ, \quad \angle YXW = \frac{B}{3} + 60^\circ.$$

Let Y′ and Z′ be the reflections of Y and Z in XW and XV, respectively. Then V, Y′, Z′, W are collinear; for

$$\angle Z'XY' = \angle ZXZ' + \angle YXY' + 60^\circ - 360^\circ = 60^\circ - \frac{2A}{3}$$

and consequently each of the base angles of the isosceles triangle Z′XY′ is equal to $60^\circ + \frac{A}{3}$.

Thus $$\angle XVW = \frac{B}{3}, \qquad \angle XWV = \frac{C}{3}.$$

Thus the theorem holds for triangle UVW and therefore also for the given triangle ABC which is equiangular to UVW.

Case (ii) *External Trisectors.*

The construction and proof are similar to *Case* (i).

Question 1623.

(HANSRAJ GUPTA) :—Prove that

$$\frac{1!\,2!\,3!\,4!\,\ldots\,(r-1)!\,(nr)!}{n!\,(n+1)!\,(n+2)!\,\ldots\,(n+r-1)!}$$

is an integer for all positive integral values of n and r.

Solution by the Proposer.

Let p be any prime number. Then, since the number of times that p occurs as a factor of $q!$ is $\sum_{k=1}^{\infty}\left[\frac{q}{p^k}\right]$ where $[x]$ denotes the greatest integer in x, it follows that p occurs as a factor of the numerator

$$\sum_{k=1}\left\{\left[\frac{1}{p^k}\right]+\left[\frac{2}{p^k}\right]+\ldots+\left[\frac{r-1}{p^k}\right]+\left[\frac{nr}{p^k}\right]\right\}$$

$$\equiv \sum_{k=1}\{\mu(p^k)\} = S_1 \text{ times.}$$

It occurs as a factor of the denominator

$$\sum_{k=1}\left\{\left[\frac{n}{p^k}\right]+\left[\frac{n+1}{p^k}\right]+\left[\frac{n+2}{p^k}\right]+\ldots+\left[\frac{n+r+1}{p^k}\right]\right\}$$

$$\equiv \sum_{k=1}\{\delta(p^k)\} = S_2 \text{ times.}$$

We proceed to prove that $S_1 \geqslant S_2$, for all values of p.

Let m be any number of the form p^k. [This restriction is not however essential.]

Let $\qquad n \equiv am + n_1, \qquad$ where $\qquad 0 \leqslant n_1 < m.$

Then,

$$\mu(m) = \left[\frac{1}{m}\right]+\left[\frac{2}{m}\right]+\left[\frac{3}{m}\right]+\ldots+\left[\frac{r-1}{m}\right]+ar+\left[\frac{n_1 r}{m}\right]$$

$$= \gamma(m) + ar + \left[\frac{n_1 r}{m}\right]$$

and

$$\delta(m) = ar + \left[\frac{n_1}{m}\right] + \left[\frac{n_1+1}{m}\right] + \left[\frac{n_1+2}{m}\right] + \ldots + \left[\frac{n_1+r-1}{m}\right].$$

Case 1. $m \geqslant r$, so that $\gamma(m) = 0$.

(i) If $n_1 < m-(r-1)$, then $\mu(m) = ar + \left[\frac{n_1 r}{m}\right] \geqslant ar \geqslant \delta(m)$.

(ii) If $n_1 = m-(r-k)$, $k < r$, then $\delta(m) = ar + k$;

$$\text{and } \mu(m) = ar + \left[\frac{(m-r+k)\,r}{m}\right],$$

$$= ar + \left[\frac{km + (m-r)(r-k)}{m}\right],$$

$$= ar + k + \left[\frac{(m-r)(r-k)}{m}\right].$$

$\therefore \quad \mu(m) \geqslant \delta(m)$.

Case 2. $m < r$.

$$\text{Then } \mu(m) = \left[\frac{1}{m}\right] + \left[\frac{2}{m}\right] + \ldots + \left[\frac{r-1}{m}\right] + \left[\frac{nr}{m}\right]$$

$$= \gamma(m) + ar + \left[\frac{rn_1}{m}\right]$$

$$= ar + \gamma(m) + \left\{ \geqslant \left[\frac{r}{m}\right] + \ldots + \left[\frac{r+n_1-1}{m}\right] \right\}, \text{ since } m < n_1.$$

$$\geqslant ar + \left[\frac{1}{m}\right] + \left[\frac{2}{m}\right] + \left[\frac{3}{m}\right] + \ldots + \left[\frac{r+n_1-1}{m}\right];$$

$$\geqslant ar + \left[\frac{n_1}{m}\right] + \left[\frac{n_1+1}{m}\right] + \ldots + \left[\frac{n_1+r-1}{m}\right].$$

$$\geqslant \delta(m)$$

in each case. Hence $\mu(m) \geqslant \delta(m)$, and $S_1 \geqslant S_2$ which proves the result.

Questions 1632, 1633.

(R. VAIDYANATHASWAMY):—ABCD is a cyclic quadrangle. If the pedal line of A with respect to BCD is perpendicular to the Euler line (*i.e.*, join of circum-centre and centroid) of BCD, prove that a similar property holds good for each of the vertices.

When this property is possessed by ABCD, prove that the perpendiculars let fall from each vertex on its pedal line with respect to the remaining three concur in a point.

(R. VAIDYANATHASWAMY):—Prove that

$$\tan\frac{a+b+c-d}{2}+\frac{\cos a+\cos b+\cos c}{\sin a+\sin b+\sin c}=0$$

is a symmetric relation between a, b, c, d. Prove furthur that the two relations,

$$\tan\frac{a+b+c+d+\pi-2e}{2}+\frac{\cos a+\cos b+\cos c+\cos d}{\sin a+\sin b+\sin c+\sin d}=0$$

and the similar equation in which d and e are interchanged, constitute two symmetric conditions on a, b, c, d, e.

Indicate the generalisation to n numbers $a, b,$.........

Solutions by K Rangaswami.

(i) The first question is a consequence of the known theorem, *viz.*, If a rectangular hyperbola passes through the vertices of a triangle and its circum-centre, it cuts the circum-circle again at the point diametrically opposite to the Euler point* of the triangle.

Given that the pedal line of D with respect to ABC is perpendicular to the Euler line of ABC, the point D is uniquely fixed on the circum circle Ω of ABC as the point diametrically opposite to the Euler point of ABC. From the theorem above stated, the unique rectangular hyperbola Γ through ABC and O, the centre of Ω, passes also through D and is thus the unique conic in the pencil [ABCD] outpolar to Ω. It is in fact the isogonal transformation of the Euler line of ABC and is called the Jerabek's† hyperbola of the triangle.

* The Euler point of a triangle is the point whose pedal line with respect to the triangle is parallel to the Euler line of the triangle.

† *Vide* Casey: ***Analytical Geometry***, p. 420.

We may also note that D is the isogonal conjugate of the point at infinity on the Euler line and that Γ passes through the symmedian point of ABC.

The symmetry of the result follows from the fact that Γ circumscribes each of the four triangles and passes through their circum-centre; so that each of the four points is diametrically opposite to the Euler point of the triangle formed by the other three.

If H_a, H_b, H_c, H_d be the ortho-centres of the triangles BCD, etc., AH_a, etc., passes through the centre K of Γ on the nine point circle of ABC. K is also the point of concurrence of the pedal lines of A with respect to BCD, etc. Hence the perpendiculars from A, B, C, D on their corresponding pedal lines concur at the point T on Γ, the *symmetrique* of O with respect to K.

(ii) A uniform proof of the two questions based on circular co-ordinates can be given as follows :—

Let Ω be a unit circle, with centre O, the points A on which are specified by circular co-ordinates

$$t = x + iy = \cos\theta + i\sin\theta\,;\ \bar{t} = x - iy = \cos\theta - i\sin\theta$$

referred to O. Let the suffix r refer to the point A_r and let θ_r be the angular co-ordinate of A_r measured from a line OX with O as pole. It is seen from a known result* that the pedal line of a point A (angular co-ordinate θ_p) with respect A_1, A_2 ......, A_r makes an angle ϕ with OX given by

$$\phi = \sum_{p=1}^{r} \theta_p/2 - (r-2)\,\theta/2 + (r-)\,\pi/2 \qquad \dots \quad (1)$$

Expressing the condition that in a set of r points the pedal line of A with respect to the remaining $(r-1)$ points should be perpendicular to the direction OK $[K \equiv \sum_1^r t_{\cdot}, \sum_1^r \bar{t}_1]$ we get the relation

$$\left(\frac{1}{t_1} + \dots + \frac{1}{t_{s-1}} + \frac{1}{t_{s+1}} + \dots \frac{1}{t_r}\right) t_1 t_2 \dots t_{s-1} t_{s+1} \dots t_r$$
$$+ (-)^r t_s^{r-3} (t_1 + \dots t_{s-1} + t_{s+1} + \dots + t_r) = 0. \qquad (2)$$

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* *J. I. M. S.*, Vol. 19, Part II, 1932, p. 269.

whose equivalent in terms of θ is

$$\tan\frac{\theta_1+\ldots+\theta_{s-1}+\theta_{s+1}+\ldots\theta_r-(r-3)\,\theta_s+(r-4)\pi}{2}$$

$$+\frac{\cos\theta_1+\ldots+\cos\theta_{s-1}+\cos\theta_{s+1}+\ldots+\cos\theta_r}{\sin\theta_1+\ldots+\sin\theta_{s-1}+\sin\theta_{s+1}+\ldots+\sin\theta_r}=0. \quad (3)$$

If, in equations (2) and (3), we keep the indices 1, 2, 3 fixed and interchange the others we get $(r-3)$ equations of the form

$$\sigma_{r-2}-\sigma_{r-3}\,t_s+\ldots+(-)^{r-4}\,\sigma_2\,t_s^{r-4}=0 \qquad \ldots \quad (4)$$

where σ_p is the product of the t's taken p at a time) of degree $(r-4)$ in t's; and hence (4) is satisfied by $t_1, t_2, \ldots t_r$. Hence $(r-3)$ corresponding equations (*i.e.*, of type 3) in the θ's constitute $(r-3)$ symmetric conditions on $\theta_1, \theta_2, \ldots \theta_r$, which is the required generalisation.

When $r=4$, (2) takes the form

$$\Sigma\, t_p\, t_q, = 0 \quad (p, q = 1, 2, 3, 4) \qquad \ldots \quad (5)$$

whose equivalent in terms of θ is

$$\tan\frac{\theta_1+\theta_2+\theta_3-\theta_4}{2}+\frac{\cos\theta_1+\cos\theta_2+\cos\theta_3}{\sin\theta_1+\sin\theta_2+\sin\theta_3}=0 \qquad \ldots \quad (6)$$

which proves the first part of the two questions.

Q. 1632 solved analytically by S. Srinivasan and by the use of isotropic co-ordinates by B. Ramamurti.

RECEIVED FOR REVIEW.

Lessons in Elementary Analysis, by G. S. MAHAJANI, M.A., Ph.D. (Cantab.) [Phillip Bayliss Research Student, St. John's College, Cambridge, 1926; Life Member, D. E. Society, Fergusson College, Poona.] Second Edition, Aryabhushan Press Poona, 1934, pp. xii + 264. Price Rs. 5.

ANNOUNCEMENTS AND NEWS.

Readers are invited to contribute to the value of this section by sending suitable news items of interest.

The Silver Jubilee Commemoration Volume of the Society, the publication of which was delayed for various reasons, has now been issued, as Volume XX of the *Journal. It contains a full report of the proceedings of the Jubilee Conference* at Poona and the texts of the papers contributed thereto Copies may be had from S. Mahadeva Iyer, Esq., Assistant Professor, Presidency College, Madras, at a cost of Rs. 10 each.

The Treasurer of the Society, L. N. Subramaniam, Esq., Christian College, Madras, requests all those who have kindly promised contributions to meet the expenses of printing the Jubilee Volume, to remit the donations early to him,

Dr. S. Chowla, M.A., Ph.D., Reader, Andhra University, Waltair, Mr. K. Venkatachaliengar, M.Sc., Research Student, Indian Institute of Science, Bangalore, and Mr. V. A. Mahalingham, M.A., Lecturer, Loyola College, Madras, have been elected members of the Society.

Dr. S. C. Dhar, Professor of Mathematics, College of Science, Nagpur, has proceeded to England for prosecuting higher studies.

The *Scripta Mathematica*, Vol. I, No. 2, gives the following interesting account of the origin of the INSTITUTE FOR ADVANCED STUDY at Princeton.

In 1930 Abraham Flexner published a book entitled "Universities, American, English, German," containing a critical survey of their contributions to the world's advancement. It attracted the attention of L. Bamburger and Mrs. Felix Fuld, former owners of Bamburger and Co., who asked Dr. Flexner what kind of a centre of learning he would organise if adequate funds were placed at his disposal. After consultation with scholars throughout the world, Dr. Flexner decided to start with a School of Mathematics as the " most fundamental of all the disciplines" and for this five million dollars have been allotted. Scholarsphips and Fellowships are available for suitably qualified students without means to pursue advanced studies. Each Professor decides for himself whether to work with seminars or with each student individually, and " if a professor prefers not to give lectures or other formal instruction but to let them simply work alongside and learn as best as they can by their own efforts, he will have complete freedom to do so."

The Directorship of the School was offered to Prof. G. D. Birkhoff but he decided to remain at Harvard. This position as well as a Professorship of Mathematics and Theoretical Physics has now been accepted by Albert Einstein. Other Professors in the School are Oswald Veblen, and Herman Weyl.

Prof. G. D. Birkhoff touched Bombay on his way from Kyoto to Genoa, and was met on behalf of the Society by Prof. Kosambi and Dr. Vijayaraghavan. Prof. Birkhoff received an invitation from the President of the Indian Mathemetical Societys to extend his stay in India.

The complete works of H. A. Lorentz in nine volumes are being prepared for publication under the editorship of P. Zeeman, P Ehrenfest and A. D. Fokker. The Publisher is Martinus Nigoff, of the Hague.

The American Mathematical Society has recently issued the sixteenth volume of its Colloquium publications, *Algebraic Functions* by G. A. Bliss. Price $ 3.00.

A new Journal devoted exclusively to the Theory of Numbers is to be published under the title *Acta Arithmetica*. The first issue is expected to appear in November. The Editorial Board consists of Drs Walfisz Levelski, and Dickstein.

An international journal. *Compositio Mathematica*, is to be published by Noordhoff of Groningen, the Board of Editors consisting of Bieberbach, Brouwer, Donder, Julia and Wilson.

The University of Moscow publishes a Journal entitled "Abhandlungen au dem Seminar fur Vector und Tensoranalysis samt Anwendungen and Geometrie, Mechanik, and Physik" under the editorship of B. Kagan.

We are glad to announce that *The Scripta Mathematica*, a quarterly journal devoted to Philosophy, History, and expository treatment of Mathematics, is being received in exchange by the Society.

The Mathematics Teacher, official organ of the National Council of Teachers of Mathematics, is now being received in exchange for *The Mathematics Student*

Prof. E. H. Neville, of the University of Reading, has been elected President of the (British) Mathematical Association.

Prof. G. N. Watson has been elected President of the London Mathematical Society; Vice-Presidents: Professors A. G. Dixon, G. H. Hardy and G. F. J. Temple.

The University of Madras has decided to confer the degree of Doctor of Science on S. Sivasankaranarayana Pillai, B.A., M.Sc., Lecturer in Mathematics in the Annamalai University. The Thesis submitted for the degree relates to certian order results in the theory of numbers and to numbers analogous to Ramanujan's Highly Composite Numbers

Rao Bahadur P. V. Seshu Aiyar has accepted an invitation from the Annamalai University to deliver a course of six lectures on the Theory of Sets of Points. The lectures commence on the 18th October and are open to all who wish to attend.

The Comstock Prize of 2,500 dollars was presented by the National Academy of Sciences to Dr. P. W. Bridgman, Professor of Mathematics and Philosophy at Harvard, in recognition of his brilliant achievements in advancing our knowledge of the behaviour of matter.

Mr. G. P. Rao of the Survey of India, Dehra Dun, died on the 7th Jnly 1933.

St. Andrews Colloquium.

Prof. Dhar writes from Edinburgh:

Under the auspices of the Edinburgh Mathematical Society, a Colloquium was held at St. Andrews from the 18th July for ten days. A number of distinguished

mathematicians assembled from all parts of the world; notable amongst them being Profs. De Sitter, E. T. Whittaker, E. A. Milne, Brown, and several others.

Prof. H. W Turnbull, F.R.S., of the University of St. Andrews, was the President. The Secretaries were Dr. I M. H. Etherington, Dr. G. C. McVittie, and Dr. D. E. Rutherford. The following were the lectures delivered :

1. Prof. E. A. MILNE, F.R.S. (Oxford) on "World-Structure by Kinemetic Methods of Special Relativity" (six lectures).

2. Prof. H. W. TURNBULL, F.R.S. (St. Andrews) on "Pictorial Geometry."

3. Prof. B. M. WILSON (Dundee) on "Ramanujan's Note-books and their place in Modern Mathematics" (two lectures).

4. Mr. W. L. FERRAR (Oxford) on "Some Expansions relating to the Problem of Lattice Points" (two lectures).

Prof. B. M. Wilson spoke about Ramanujan and his work in very high eulogising terms, quoting what Prof. Hardy had written about him in the Proceedings of the London Mathematical Society. He and Prof. Watson have undertaken to edit Ramanujan's work and it is expected it will take fifteen years to edit it. He delivered two lectures on Ramanujan's work. Prof. W. De Sitter gave a discourse on the expanding Universe at the Colloquium. Prof. J. M. Whittaker gave a lecture on "Integral Functions."

There were lectures and discussions on several other topics.

The Indian Academy of Sciences, Bangalore.

The first general meeting of the Indian Academy of Sciences was held in the Institute of Science on the 31st of July 1934, when Sir Mirza M. Ismail, Dewan of Mysore, inaugurated the Academy. Many distinguished scientists from different parts of India and visitors were present. At the business meeting held the same day, Sir C. V. Raman was elected as the President of the Academy. Rules and regulations for the conduct of the Academy were passed. An influential council was appointed with Rao Bahadur Prof. B. Venkatesachar and Prof. C. R. Narayana Rao as Secretaries.

The first number of the Proceedings of the Academy was issued to the Fellows the same day.

The 1st and 2nd of August were spent in the discussion of many papers presented to the Academy. A discussion on molecular spectra was led on the 3rd by Prof. Samuel of the Aligarh University and a few papers of interest on the same subject were read on the 4th of August.

Business meetings of the Academy will be held once every month when papers contributed will be read and discussed. An annual meeting will be held in July every year.

ACTIVITIES OF STUDENTS' ASSOCIATIONS.

University of Dacca.

Dr. M. Basu writes :—The University Mathematics Seminar meets once a week and Honours and post-graduate students are encouraged to speak. Last session the following First Year Honours students spoke on the subjects mentioned.

M. C. Chakravarty :	" On Transcendental Numbers."
Nawab Ali Meah :	" Amsler's Planimeter."
Manindra Kumar Roy :	" Legendre's Polynomials."
A. K. Paul :	" On Gamma Functions."

Loyola College Mathematics Association.

Leecturer.		Subject.
Dr. R. Vaidyanathaswamy, D.Sc	..	Time and the Theory of Relativity
H. H. Jagadguru Sri Sankaracharya of Puri Govardhan Mutt		Mathematics and other Positive Sciences, Ancient and Modern.
Rao Bahadur P. V. Seshu Ayyar, B.A., L.T.		The Nature of Mathematics and its relation to other Sciences.
Mr. M. Narayanamurti, M.A.	...	Wireless.
Mr. P. V. Natesan, M.A.	...	The Theory of the Simple Voltaic Cell.
Mr. M. Narayanamurti, M.A.	...	Wireless.
Dr. H. Parameswaran, M.A., D.Sc.	...	Technical Education in India.
Mr. A. Somayajulu, M.A.	...	Some Aspects of Hindu Mathematics

Annamalai University Mathematics Association.

The following papers were read during the academic year 1933-34 :—

P. Jagannadhan	...	Class V Hons.	...	" Waring's Probem."
V. T. Srinivasan	...	"	...	" Transfinite Nnmbers."
N. Ramachandran	...	"	...	" Transcendence of e."
A. N. Sankaran	...	"	...	" Calculus of Finite Differences "
S. Ramaswamy	...	"	...	" Vectorial mechanics."
R. Subrahmanyam	...	"	...	" Number, the Language of Mathematics."
V. Seetharaman	...	"	...	" The curve of quickest descent under gravity."
S. Pichai	...	"	...	" Theory of sets of points."
K. Rangaswamy, B.A.		Hons. Research Student.	...	" The infinite Regions of various geometries."

QUESTIONS FOR SOLUTION.

Proposers of Questions are requested to send their own Solutions along with their Questions.

1677. (ALFRED MOESSNER):—We have

$$7^n + 8^n = 1^n + 5^n + 9^n \qquad (n = 1, 3).$$

Is the equation $\quad M_1^n + M_2^n = N_1^n + N_2^n + N_3^n \qquad (n = 1, 3)$
also solvable when $M_1 = M_2$?

1678. (HANSRAJ GUPTA):—If $|x| < 1$, then

$$\frac{1}{(1-x^2)(1-x^3)(1-x^4)\ldots} = 1 + \sum_{s=1}^{\infty} \frac{x^{s^2}}{(1-x)(1-x^2)\ldots(1-x^s)}$$

1679. (RATAN LAL GUPTA):—Show that

$$\text{(i)} \quad \int_0^y \frac{x \sin ax}{\sqrt{y^2 - x^2}}\, dx = \frac{\pi}{2}\, y\, J_1(ay)$$

$$\text{(ii)} \quad \int_0^{\pi/2} J_1(y \sin \theta)\, d\theta = \frac{1 - \cos y}{y}$$

where J_1 is the Bessel function of the first order.

1680. (K. SATYANARAYANA):—A pair of real isogonal conjugates with respect to a given triangle exists on a given line when and only when the line cuts a pair of opposite sides of the ortho-centric tetrad (formed by the in- and ex- centres) one internally and one externally.

1681. (A. A. KRISHNASWAMI AYYANGAR):—If the F-conic of two circles degenerates, show that it is either a pair of parallel straight lines or a pair of perpendicular straight lines; and in the latter case, show that the perpendicular lines meet at one of the limiting points of the co-axal system determined by the circles, the other limiting point being the centre of the degenerate ϕ-conic.

1682. (K. RANGASWAMI):—The feet of the normals from a point P to a given conic S lie on a rectangular hyperbola H through P. Show that if H passes through a fixed point Q in the plane, the locus of P is a straight line l through Q.

Show also that if Q moves along a diameter of S, l remains parallel to itself.

// THE MATHEMATICS STUDENT

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ON CIRCULAR CYLINDERS CIRCUMSCRIBING A TETRAHEDRON.

By V. RAMASWAMI AIYAR.

1. The problem of determining all the circular cylinders circumscribing a tetrahedron ABCD occurred to the writer in a certain connection, but proved to be a rather tough problem. The writer was not able, by methods of pure geometry, to reduce it to known theorems. But the theorem underlying the problem was soon spotted and turned into other geometrical forms in which it could be more easily tackled analytically. This analysis was tough work and some little smartness was required for carrying it out successfully at a point where matters looked hopeless. It is proposed to give here a general account of this research ; and the writer hopes that some interested reader may be able to find out a comparatively simpler proof of the underlying theorem.

2. With reference to a tetrahedron ABCD, by "isogonal conjugates" we shall mean a pair of points P, P′ such that the normal co-ordinates of one are reciprocals of the normal co-ordinates of the other. The principal theorem can now be stated thus:—

If O be a point at infinity whose isogonal conjugate O′ also lies at infinity, then, the lines AO, BO, CO, DO are generators of a circular cylinder ... (2.1)

It follows that AO′, BO′, CO′, DO′ are also generators of a circular cylinder. The circumscribing circular cylinders thus occur in pairs.

3. Let the areas of the faces of the tetrahedron be denoted by A, B, C, D. If a point O, of normal co-odinates α, β, γ, δ lies at infinity, we have

$$A\alpha + B\beta + C\gamma + D\delta = 0 \qquad \ldots \quad (3.1)$$

Its isogonal conjugate O′ will be the point $1/\alpha$, $1/\beta$, $1/\gamma$, $1/\delta$. If this also lies at infinity, we should have

$$\frac{A}{\alpha}+\frac{B}{\beta}+\frac{C}{\gamma}+\frac{D}{\delta}=0;$$

that is,

$$A\beta\gamma\delta + B\gamma\delta\alpha + C\delta\alpha\beta + D\alpha\beta\gamma = 0 \qquad \ldots \quad (3.2)$$

The point O is thus any point of the cubic curve, Γ, in which the cubic surface (3.2) cuts the plane at infinity (3.1).

Taking now a fixed point P, if we draw lines through P to all the points of the cubic curve Γ, we shall have a cubic cone, Γ_P, of vertex P; and we infer that the axes of circular cylinders circumscribing ABCD are parallel to generators of the cubic cone Γ_P.

4. Another way of stating the underlying theorem is as follows:—

Taking a sphere, centre at any point P, let α, β, γ, δ be great circles of the sphere whose planes are parallel to the faces of the tetrahedron ABCD.

If now O, O′ are foci of any sphero-conic inscribed in the spherical quadrilateral formed by α, β, γ, δ, then, lines drawn through the vertices of the tetrahedron ABCD parallel to PO (or parallel to PO′) are generators of a circular cylinder ... (4.1)

Yet another way of stating the theorem is as follows:—

Through any point P draw planes α, β, γ, δ parallel to the faces of the tetrahedron ABCD. If now any point O is such that the feet of the perpendiculars from it on α, β, γ, δ lie in a plane X, then, lines drawn through A, B, C, D parallel to PO are generators of a circular cylinder ... (4.2)

We also have the following:—

The perpendiculars drawn through A, B, C, D to the plane X are generators of a circular cylinder ... (4.3)

The form (4.2) is perhaps the simplest form of stating the underlying theorem.

5. There is a theorem worth noting that forms the connecting link between the different forms of the underlying theorem given in the last para.

Let P be the vertex and PL, PL′ the focal lines of a quadric cone V. *Then, if S be any point in PL, the locus of the feet of the perpendiculars from S on tangent planes to the cone V is a circle X, whose plane is perpendicular to the focal line PL′* ... (5.1)

This may be called the pedal circle of S.

We add the following for enabling a fuller view of this theorem. If S′ be a point in PL′, the locus of the feet of the perpendiculars drawn from S′ on tangent planes to the cone V is a circle X′, whose plane is perpendicular to PL. This is the pedal circle of S′. Then we have the theorem that

The pedal circles X and X′ lie on a sphere whose centre is the middle point of SS′ ... (5.2)

6. We proceed now to some detailed consideration of the cubic curve Γ at infinity and the corresponding cubic cone Γ_P of vertex P.

The cubic Γ, as we saw, consists of points O, O′, which are isogonal conjugates and will generally be distinct. The cubic will usually be a bipartite or unipartite curve, and *not have* a double point. But if the faces of the tetrahedron are connected by one of the relations

$$D + A = B + C \qquad \ldots \quad (6.1)$$

$$D + B = C + A \qquad \ldots \quad (6.2)$$

$$D + C = A + B \qquad \ldots \quad (6.3)$$

then a double point occurs on the cubic by the coincidence of one pair of isogonal conjugates on it. Thus, in the case (6.1), the double point is $(1, -1, -1, 1)$, in the case (6.2), it is $(-1, 1, -1, 1)$, and in (6.3), it is $(-1, -1, 1, 1)$. *The condition for the occurrence of a double point on the cubic* Γ is really

$$U \equiv (D + A - B - C)(D + B - C - A)(D + C - A - B) = 0 \qquad \ldots \quad (6.4)$$

Though in U, as above written, there is a special treatment of D, yet this is only apparent, and the function is symmetric in A, B, C, D.

Pursuing the point further, it may be deduced that *the condition for the curve Γ to be unipartite is $U > 0$; and the condition for its being bipartite is $U < 0$* ... (6.5)

Correspondingly, the cone Γ_P will be *unipartite* if $U > 0$; *bipartite*, if $U < 0$, and *nodal* if $U = 0$ ... (6.6)

7. If the tetrahedron be such that two of the conditions (6.1), (6.2), *(6.3) hold simultaneously, then the cubic Γ will have two double points ;* which means, it breaks up into the line at infinity joining these points and a conic at infinity passing through them. The cubic cone Γ_P correspondingly breaks into a plane and a cone of degree 2.

Thus if the two conditions holding be (6.2) and (6.3), that is,

$$\left.\begin{array}{l} D + B = C + A \\ D + C = A + B \end{array}\right\}$$

these give $B = C$ and $A = D$. Using these in para 3, we easily get at the following result :—

If, in the tetrahedron ABCD, the faces meeting along the edge AD are equal in area and likewise those meeting along the edge BC, then, the generators of circular cylinders circumscribing the tetrahedron fall into two sets ; in one set the generators are parallel to the medial plane of the tetrahedron which bisects the edges AB, AC, DB, BC ; in the other set, the generators are parallel to generators of the asymptotic cone of the hyperboloid $\alpha\delta = \beta\gamma$ ... (7.1)

8. If the tetrahedron be such that all the three conditions (6.1), (6,2), (6.3) hold, we should have $A = B = C = D$. When the areas of the faces are equal, the tetrahedron should be isosceles, that is, one whose opposite edges are equal. In this case, then, Γ breaks up into three lines at infinity and Γ_P breaks up into three planes ; and we have the following result :

The generators of circular cylinders circumscribing an isosceles tetrahedron fall into three sets, the generators in each set being parallel to one of the medial planes of the tetrahedron ... (8.1)

The reader may find that the three medial planes of an isosceles tetrahedron are perpendicular to one another.

9. Let A, B, C, D be four co-planar points forming the vertices of a convex quadrangle ABCD. Let us seek to know all the circular

cylinders circumscribing the quadrangle. If we treat the quadrangle as an extreme case of a tetrahedron, the generators are parallel, as we saw, to those of a cubic cone. But, drawing parallels in any direction in the plane of the quadrangle itself, these can be regarded as generators of an immensely big circular cylinder constituted by the plane itself. Hence the cubic cone of the general case should break up into a plane (parallel to the plane of the quadrangle) and a cone of the second degree. Deleting the former, we see that

The generators of circular cylinders passing through four coplanar points forming a convex quadrangle are parallel to the generators of a cone of the second degree ... (9.1)

A special case arises when A, B, C, D are four concyclic points. In this case the cone of the second degree breaks up into two planes. Let the diagonals AC, BD intersect in O ; and let OX, OY bisect the angles at O made by them. Draw OZ perpendicular to the plane of ABCD. We have the result :

The generators of circular cylinders circumscribing the four concylic points A, B, C, D fall into two sets ; one set being lines parallel to the plane XOZ, and the other set being parallel to the plane YOZ ... (9.2)

10. A notable point both in connection with a tetrahedron ABCD and a convex quadrangle ABCD is as follows :

For every tetrahedron ABCD we can find an infinite number of others A′B′C′D′, of different forms, such that for any circular cylinder circumscribing the one there are circular cylinders, **with axes in the same direction,** *circumscribing the others* ... (10.1)

A like property holds for quadrangles also. In particular,

For every convex quadrangle ABCD in a plane we can assign a rhombus *A′B′C′D′ in the plane, such that for any circular cylinder circumscribing the one, there is a circular cylinder, with axis in the same direction, circumscribing the other* ... (10.2)

In this case, if A, B, C, D be concyclic, the rhombus A′B′C′D′ becomes a square.

CONTINUOUS GROUPS AND TWO THEOREMS OF EULER.

By D. D. Kosambi.

In the present note, I show how the theory of continuous groups is applicable to the discussion of Euler's theorem (1755) on homogeneous functions and to the equation of continuity in hydro-dynamics, also due to Euler.

All operations take place in a continuous manifold of the n variables $x^1, x^2, \dots x^n$. The notation $f_{,i} = \partial f / \partial x^i$ is used for partial differentiation, along with the tensor summation convention for a repeated or "dummy" index : $A^r B_r = A^1 B_1 + A^2 B_2 + \dots + A^n B_n$. The first section gives known results which are used as lemmas in the other two sections.

1. A continuous one parameter group of transformations is said to be generated by a vector field $u^i(x)$ or by the operator $D = u^r \partial / \partial x^r$ or by the infinitesimal transformation $\bar{x}^i = x^i + u^i \delta\tau$. The finite transformations in terms of a parameter t are given by

$$\bar{x}^i = x^i + t u^i + \frac{t^2}{2!} D u^i + \dots \frac{t^n}{n!} D^{n-1} u^i + \dots = \sum_{k=0}^{\infty} \frac{t^k}{k!} D^k (x^i) \quad \dots \quad (1)$$

We shall have to distinguish between such transformations and point-transformations or changes of co-ordinates denoted by $x \longleftrightarrow x'$. A function of the co-ordinates that retains its value at corresponding points under non-singular point transformations is called an absolute invariant, or tensor of rank zero. A contravariant or velocity-like vector λ^i and a covariant or force-like vector μ_i transform thus :

$$\lambda'^i = \lambda^r \frac{\partial x'^i}{\partial x^r} \qquad \mu'_i = \mu_r \frac{\partial x^r}{\partial x'^i} \qquad \left| \frac{\partial x'^i}{\partial x^j} \right| \neq 0 \quad \dots \quad (2)$$

Sets of functions that behave like the algebraic product of any number of vectors of each kind are called tensors. Some functions,

however, cannot be brought under this scheme, having the transformation :

$$F(x') = F(x) \left\{ \left| \frac{\partial x'^i}{\partial x^j} \right| \right\}^p \qquad \dots \quad (3)$$

under change of co-ordinates. These are called relative invariants of weight p.

The generators u^i of the one parameter group (1) form a contravariant vector field. Absolute invariants have the law of transformation under (1) given by

$$\bar{f} = f + t\,Df + \frac{t^2}{2!} D^2 f + \dots = \sum_{k=0}^{\infty} \frac{t^k}{k!} D^k f \qquad \dots \quad (4)$$

If $\bar{f} = f$, *i.e.*, $Df = 0$, f is said to be an *invariant of the group.* We shall consider relative invariants in the third section.

2. If under the group (1), a function is transformed as

$$\bar{f} = \psi(f)\,\phi(t) \qquad \dots \quad (5)$$

we shall say that the function has a multiplicative transform. With this definition, it is obvious that

The only multiplicative transforms are of type $\bar{f} = fe^{ct}$. *A necessary and sufficient condition thereto is that* f *be a solution of*

$$Df = cf \qquad \dots \quad (6)$$

Moreover,

The only groups under which the co-ordinates themselves have multiplicative transforms are those given by

$$\bar{x}^i = x^i e^{c_i t}.$$

The two results can be combined. After putting $\lambda = e^t$, we have $f(\lambda c_1 x^1 \dots \lambda c_n x^n) = \psi(f)\,\phi(\lambda)$ *is possible only when* $\psi\phi = \lambda^c f$. *A necessary and sufficient condition thereto is*

$$c_1 x^1 \frac{\partial f}{\partial x^1} + c_2 x^2 \frac{\partial f}{\partial x^2} + \dots + c_n x^n \frac{\partial f}{\partial x^n} = cf.$$

For $c_1 = c_2 = c_3 = \dots = c_n$, this is Euler's theorem on homogeneous functions. Let us attempt another generalization by searching

for functions with transforms under (1) of the type $\bar{f} = \psi(f, t)$ Thereby, $\psi(f, 0) = f$. Also, $\psi_t(f, 0) = Df$ must be a function of f alone: $Df = \phi(f)$. Consider the function H, defined by $H'\phi = H$. Then $H(f)$ has necessarily the transform

$$\bar{H} = He^t \qquad \phi(f) = Df = \frac{H}{H'} \neq 0 \qquad \dots \quad (7)$$

Therefore, *If a transform of the type* $\bar{f} = \psi(f, t)$ *is possible, then either f is an invariant of the group,* $(\psi_t = 0)$, *or there exist functions of f with a multiplicative transform, one such being given* by $H(f)$ where $\log H(y) = \int dy / \phi(y)$. $\phi(f) = Df \neq 0$.

For Euler's theorem, the corresponding result naturally reads:

If $f(\lambda x) = \phi(f, \lambda)$ *is possible, either f is homogeneous of degree null in* x^i *or there exist functions of f homogeneous of any degree. One of the first degree, moreover, is given by* $H(f)$ *where* $H/H' = Df \neq 0$.

This is about as much as can be done to generalise either the concept of homogeneous functions, or Euler's theorem. Here, Df has the special form $x^r f_{,r}$.

3. We come now to group transformation of some functions that are relative invariants under point transformation. The fundamental theorem is:

The Jacobian of the finite transformation (1) *is given by*

$$J = \left| \frac{\partial \bar{x}^i}{\partial x^j} \right| = 1 + \theta t + \frac{t^2}{2!}(D\theta + \theta^2) + \dots = \sum_{k=0}^{\infty} \frac{t^k}{k!} \Theta^k (1) \dots \quad (8)$$

where $\theta = u^r_{,r}$ *and** $\Theta = D + \theta$.

Herewith, the necessary convergence and differentiability conditions must be assumed. The theorem may be proved by direct computation, though I give another proof by induction.

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* Θ^k does not indicate the binomial expansion of $\{D + \theta\}$, but a k-fold repetition of the operation Θ.

For $n > 1$, the dimensions of the Jacobian can be reduced by unity by change of co-ordinates in such a fashion that the lines of flow lie on the surfaces (are characteristics of) $x'^n = \text{const.}$ and $|\partial x'^i / \partial x^j| = 1$. This can always be done. In these new co-ordinates, $u'^n = 0$; $x'^n = x^n$ and the Jacobian is a determinant of the same type, but of order $n-1$. The one change is that ordinary partial differentiation must be replaced by co-variant differentiation. As higher co-variant derivatives are not commutative in general, because of

$$u^i_{/j/k} - u^i_{/k/j} = R^i_{jkl}\, u^l \qquad \dots \quad (9)$$

we must, for the purpose of the deduction only, regard the manifold of the x^i as Euclidean. Then (8) will be in the invariantive form under change of co-ordinates, the theorem true for $n = m+1$ if true for $n = m$. But for $n = 1$, the theorem is almost obvious, and very easily proved; hence, for all n. It must again be empbasized that (8) is to be regarded as a purely analytical formula, quite apart from the geometry that may afterwards be assigned to the manifold.

It is clear that $dJ/dt = \Theta J$. Therefore, we may state a corollary:

$$J = \sum_{k=0}^{\infty} \frac{t^k}{k!} \{ D + p\Theta \}^k (1) = 1 + \sum_{k=1}^{\infty} \frac{t^k}{k!} \{ D + p\Theta \}^{k-1} (p\Theta) \qquad \dots \quad (10)$$

A second theorem is very easily proved as before, or directly.

If ρ transforms as in (4), *then*

$$\bar{\rho} J = \rho + t(D\rho + \rho\Theta) + \dots = \sum_{k=0}^{\infty} \frac{t^k}{k!} \Theta^k \rho \qquad \dots \quad (11)$$

The best application is, of course, in hydrodynamics. The generators u^i are interpreted as the velocity-components of the fluid, and ρ the density, the entire movement being steady. For the conservation of mass of every portion of the fluid, we must have $\bar{\rho} J = \rho$. This leads to the result:

A necessary and sufficient condition for the continuity of fluid motion is that $\Theta\,\rho \equiv D\rho + \rho u^r{}_{,r} = 0.$

If this is satisfied initially, throughout the fluid, it will be satisfied everywhere for all time, and the motion which is continuous at the start in the sense of the conservation of mass, will always remain so continuous.

The case where t enters explicitly in the field of velocities, and the movement consequently depends on the time, can always be brought under the preceding by regarding t as an additional co-ordinate x^0, with $u^0 = 1$. The parameter of the group being labelled s, we have

$$\bar{t} = t + s \qquad \ldots \quad (12)$$

$$D = \partial / \partial t + u^r \partial / \partial x^r$$

$$\Theta = u^r{}_{,r}$$

$$J = \partial (\bar{x}, \bar{t}) / \partial (x, t) = \partial (\bar{x}) / \partial (x) \qquad \text{as before.}$$

The previous expansion holds with the addition of a term $\partial / \partial t$ to D. The necessary and sufficient condition for continuity is

$$\frac{\partial \rho}{\partial t} + u^r \frac{\partial \rho}{\partial x^r} + \rho u^r{}_{,r} = 0 \qquad \ldots \quad (13)$$

which must now be fulfilled everywhere and for all time.

The same problem for Riemannian spaces is very easily solved, and of some importance in the theory of relativity. If $\mu^2 = \pm \left| g_{ij} \right| \neq 0$ be the discriminant of the space, the conservation of mass comes to

$$\bar{\rho}\, \bar{\mu} J = \rho \mu \qquad (14)$$

From the preceding,

$$\bar{\rho}\, \bar{\mu}\, J = \sum_{k=0}^{\infty} \frac{t^k}{k!} \Theta^k (\rho\mu) \qquad (15)$$

Here, Θ is still to be computed with ordinary partial derivatives and therefore not in a form independent of the choice of co-ordinates.

However, ρ, being an absolute invariant, has the same partial and co-variant derivative: $\rho_{,i} = \rho_{/i}$. From the well-known properties of Christoffel symbols, we have

$$u^i_{/j} = u^i_{,j} + \Gamma^i_{jr} u^r$$
$$\mu_{,k} = \mu \Gamma^r_{rk} \qquad \dots \quad (16)$$

Therefore,

$$\Theta \rho\mu = \{ \mu u^r \rho_{,r} + \rho u^r \mu_{,r} + \rho \mu u_{,r}^r \} = \mu \{ u^r \rho_{/r} + \rho u^r_{/r} \} \dots (17)$$

That is, the operation Θ performed upon μ times an absolute invariant, gives μ times another absolute invariant, as the operator represented by $\Theta' f = u^r f_{/r} + f u^r_{/r}$ is of the invariantive sort.

Consequently,

$$\bar{\rho}\,\bar{\mu}\,J = \mu \sum_{k=0}^{\infty} \frac{t^k}{k!} \Theta'^k \rho \qquad \dots \quad (18)$$

The equation of continuity for Riemannian spaces can therefore be written in a form similar to the previous,

$$\Theta' \rho \equiv u^r \rho_{/r} + \rho u^r_{/r} = 0 \qquad \dots \quad (19)$$

It must be noted that the direct substitution of co-variant for partial differentiation in computing $\partial \bar{x}^i / \partial x^j$ and the subsequent calculation of J would not only be not justified, but would not give the proper result, because of the non-commutability of co-variant derivatives shown in (9).

Notes and References ·

The theorem on homogeneous functions, according to Cantor's *Vorlesungen uber die Geschichte d. Math.* appeared in Euler's book on the differential calculus in 1755. As it occupies so fundamental a position in many other subjects, it has never been thought of as a theorem in continuous groups.

The second theorem appeared in the same year in two different publications (cf. Lamb: *Hydrodynamics*, VI ed., p. 2). The treatment by means of continuous groups usually goes as far as the infinitesimal transformation, the actual expansion of the Jacobian for the finite transformation being omitted. As a result, a certain amount of confusion on the subject is to be found even in standard text-books. For instance, Appell (*Traite de Mecanique Rationelle III*, 1928, pp. 323-4) deduces the formula $dJ/dt = \theta J$ in place of $dJ/dt = \Theta J$. A somewhat different point of view is represented in Levi-Civita's *Absolute Differential Calculus*, pp. 361-2. There, one finds a treatment of a special finite transformation $\bar{x}^i = x^i + tu^i$. As a matter of fact, if this is to represent the movement of a liquid from our point of view, with t as the time and u^i the velocity field, we must have $Du^i = 0$, which gives the special case of fluid motion wherein the pressure is caused by the external force only, the movement being rectilinear and with constant velocity on each streamline. But the Jacobian of such a transformation is precisely the same as for the transformation $\bar{x}^i = x^i + tu^i + \frac{1}{2} t^2 Du^i$ provided $(Du^i),_j = 0$. Therefore, the condition that the Jacobian be identically equal to unity would represent, in general, the case of rectilinear but uniformly accelerated movement $(Du^i),_j = 0$.

The proof of formula (19) is given in detail only to show to the advanced student that an immediate substitution of co-variant for ordinary partial differentiation has to be justified.

For an absolute invariant f, we have a formula analogous to (10), *i.e.*,

$$\bar{f} = f\left\{ 1 + t\phi + \frac{t^2}{2!}(D\phi + \phi^2) + \dots \right\} = f \sum_{k=0}^{\infty} \frac{t^k}{k!} \Phi^k (1)$$

where $$\phi = \frac{Df}{f} \quad \text{and} \quad \Phi = D + \phi.$$

ON THE EQUATION

$A_1^n + A_2^n + \dots + A_s^n = B_1^n + \dots + B_s^n, \quad (n=1, 2, 3, 4, 5, 6)$

AND ALLIED FORMS.*

By Alfred Moessner in *Nurnburg, Germany.*

I. Let $$[a, a_2, \dots]^n = [b_1, b_2, \dots]^n$$
mean $$a_1^n + a_2^n + \dots = b_1^n + b_2^n + \dots$$

(a) If $a + b = c$, then we have identically

$[sa-bx, sb+cx, -sc-ax]^n = [sb-ax, -sc-bx, sa+cx]^n$, $(n=1,2,4)$
where s and x are arbitrary.

If the negative sign is not allowed, one can obtain from the above, relations of the form

(1) $[C_1, C_2, C_1 + C_2]^n = [D_1, D_2, D_1 + D_2]^n$, $(n = 2, 4)$.

For example, if we set $a = 2$, $b = 1$, $c = 3$, $s = 1$, $x = 3$, then we get

$$[-1, 10, -9]^n = [-5, -6, 11^n], \qquad (n = 1, 2, 4).$$

Hence $$[1, 10, 9]^n = [5, 6, 11]^n, \qquad (n = 2, 4).$$

Again, if (1) is true, then

$$C_1^2 + C_2^2 + C_1.C_2 = D_1^2 + D_2^2 + D_1.D_2$$

and further it follows that

$$2(C_1^2 + C_2^2 + C_1.C_2)^2 = 2(D_1^2 + D_2^2 + D_1.D_2)^2,$$
$$= [C_1, C_2, C_1 + C_2]^4 = [D_1, D_2, D_1 + D_2]^4.$$

Example. $[2, 11, 13]^n = [7, 7, 14]^n$, $(n = 2, 4)$.

Hence, $$2^2 + 11^2 + 2.11 = 7^2 + 7^2 + 7.7,$$

and $2(2^2+11^2+2.11)^2 = 2(7^2+7^2+7.7)^2 = [2, 11, 13]^4 = [7, 7, 14]^4$.

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* [Translated from the original German—Ed.].

(β) One obtains also relations of the form (1) above, by setting

$$C_1 = 4f - g; \quad C_2 = 3g - 5f; \quad -(C_1 + C_2) = f - 2g$$
$$D_1 = 2g - 5f; \quad D_2 = f + g; \quad -(D_1 + D_2) = 4f - 3g.$$

Thus $f = 3$, $g = 8$ give $[4, 9, 4 + 9]^n = [1, 11, 1 + 11]^n$ $(n = 2, 4)$.

(γ) If in (α) we choose a, b, c, s, x so that one of the terms becomes zero, then we arrive at identities of the form

$$L^n + L^n = M^n + N^n + Q^n, \qquad (n = 2, 4).$$

We can obtain such identities from simpler formulae:

$$L = f^2 + fg + g^2; \quad M = f^2 - g^2; \quad N = 2fg + g^2; \quad Q = M + N.$$

Here we have to choose f and g so that $f > g$. Thus, $f = 5$, $g = 3$, give

$$49^n + 49^n = 16^n + 39^n + (16 + 39)^n, \qquad (n = 2, 4).$$

II. If the relation

$$[C_1, C_2, C_3]^n = [D_1, D_2, D_3]^n \qquad (n = 2, 4)$$

holds then we have

$$[T - C_3, T - C_2, T - C_1, T + C_1, T + C_2, T + C_3]^n$$
$$= [T - D_3, T - D_2, T - D_1, T + D_1, T + D_2, T + D_3]^n,$$

$n = 1, 2, 3, 4, 5$, where T is arbitrary.

In this way, we arrive at identities of the form

$$\sum_{r=1}^{6} E_r^n = \sum_{r=1}^{6} F_r^n, \qquad (n = 1, 2, 3, 4, 5).$$

Example. $C_1 = 4$, $C_2 = 15$, $C_3 = 19$, $D_1 = 11$, $D_2 = 11$, $D_3 = 20$, $T = 21$, gives

$$[2, 6, 17, 25, 36, 40]^n = [1, 10, 12, 30, 32, 41]^n, \quad (n = 1, 2, 3, 4, 5).$$

III. (α) From the formula in II, we obtain the relation

$$2L^n = M^n + N^n + Q^n, \qquad (n = 2, 4);$$

or

$$[0, L, L]^n = [M, N, Q]^n, \qquad (n = 2, 4);$$

and identities of the form

$$2 \sum_{r=1}^{3} S_r^n = \sum_{r=1}^{6} V_r^n, \qquad (n = 1, 2, 3, 4, 5).$$

Example. From $2.13^n = 7^n + 8^n + 15^n$, $(n = 2, 4)$, if we put $T = 16$, we obtain the identity

$$2\,[3, 16, 29]^n = [1, 8, 9, 23, 24, 31]^n, \qquad (n = 1, 2, 3, 4, 5).$$

(β) Given the identity $2.L^n = [M, N, Q]^n$, $(n = 2, 4)$, the following relations hold good:

$$2\,L^{rn} = [L^{r-1} M, L^{r-1} N, L^{r-1} Q]^n, \; n = 2, 4 \text{ and } r = 2, 3, 4, 5, \text{ etc.}$$

Example. $2.7^n = [3, 5, 8]^n$, $(n = 2, 4)$.

Hence $2.7^{2n} = [21, 35, 56]^n$, $(n = 2, 4)$;

$2.7^{3n} = [147, 245, 392]^n$, $(n = 2, 4)$.

IV. From $[E_1, E_2, \dots E_6]^n = [F_1, F_2, \dots F_6]^n$ $(n = 1, 2, 3, 4, 5)$, it follows that

$$\sum_{r=1}^{6} \{ E_r^n + (F_r + k)^n \} = \sum_{r=1}^{6} \{ F_r^n + (E_r + k)^n \}, \; (n = 1, 2, 3, 4, 5, 6).$$

where k is any number. Thus we get identities of the form

$$[A_1, A_2, \dots A_r]^n = [B_1, B_2, \dots B_r]^n, \qquad (n = 1, 2, 3, 4, 5, 6).$$

Example.

$$[2, 3, 11, 13, 21, 22]^n = [1, 6, 7, 17, 18, 23]^n, \qquad (n = 1, 2, 3, 4, 5).$$

If we put $k = 5$, then we obtain the relation

$$[2, 3, 11, 11, 12, 13, 21, 22, 22, 28]^n = [1, 7, 7, 8, 16, 17, 18, 18, 26, 27]^n, \quad (n = 1, 2, 3, 4, 5, 6).$$

If, in the identity $2\,[S_1, S_2, S_3]^n = [V_1, V_2, \dots V_6]^n$ $(n = 1, 2, 3, 4, 5)$, we put $k = S_2 - S_1 = S_3 - S_2$, then by the above method we obtain the relation

$$[G_1, G_2, \dots G_8]^n = [H_1, H_2 \dots H_8]^n, \qquad (n = 1, 2, 3, 4, 5, 6).$$

Example. $2\,[2, 9, 16]^n = [1, 4, 6, 12, 14, 17]^n$, $(n = 1, 2, 3, 4, 5)$.

If we put $k = 9 - 2 = 16 - 9 = 7$, then we get

$$[2, 2, 8, 11, 13, 19, 21, 24]^n = [1, 4, 6, 12, 14, 17, 23, 23]^n, \quad (n = 1, 2, 3, 4, 5, 6).$$

REMARKS ON THE PREVIOUS PAPER OF DR. MOESSNER.

By S. S. PILLAI, *Annamalainagar*.

I. In the previous note, Dr. Moessner observes that, if

$$\sum_{r=1}^{6} E_r^n = \sum_{r=1}^{6} F_r^n, \qquad n = 1, 2, 3, 4, 5,$$

then it follows that

$$\sum_{r=1}^{6} \{ E_r^n + (F_r + k)^n \} = \sum_{r=1}^{6} \{ F_r^n + (E_r + k)^n \}, \quad n = 1, 2, 3, 4, 5, 6.$$

This implies the general result, namely, if

$$\Sigma\, a_t^n = \Sigma\, b_t^n, \qquad n = 1, 2, 3, \ldots, l, \qquad \ldots \quad (1)$$

then $\Sigma\, a_t^n + \Sigma\, (b_t + k)^n = \Sigma\, b_t^n + (a_t + k)^n$, $\quad n = 1, 2, \ldots, l + 1,$ $\qquad \ldots \quad (2)$

where the summation is for t from 1 to r.

This gives the interesting result that it is possible to find two sets of positive integers such that the sum, the sum of squares, the sum of cubes, ... , the sum of say, the 1000th powers of numbers in the first set are equal to the corresponding sums of the second set. For example,

$$\begin{aligned}
&(2 + {}_{1000}C_1)\, 2^n + ({}_{1000}C_1 + 2.{}_{1000}C_2 + {}_{1004}C_3)4^n \\
&\quad + ({}_{1000}C_3 + 2\,{}_{1000}C_4 + {}_{1000}C_5)6^n. \\
&\quad + ({}_{1000}C_5 + 2\,{}_{1000}C_6 + {}_{1000}C_7)8^n + \ldots \ldots \ldots \\
&= 1^n + (1 + 2).{}_{1000}C_1 + {}_{1000}C_2)3^n \\
&\quad + ({}_{1000}C_2 + 2.{}_{1000}C_3 + {}_{1000}C_4)5^n + \ldots \ldots \ldots
\end{aligned}$$

for $n = 1, 2, 3, \ldots, 1000$ and 1001.

We have got only 2^{1001} terms on each side !

Let (1) be true. In (2), put $k = 1$. Then we get

$$\Sigma\, a_t^n + \Sigma\, (b_t + 1)^n = \Sigma\, b_t^n + (a_t + 1)^n, \qquad n = 1, 2, \ldots, l + 1.$$

Apply the above result again to this by taking 1 for k.

$$\Sigma\, a_t^n + 2\, \Sigma\, (b_t + 1)^n + \Sigma\, (a_t + 2)^n = \Sigma\, b_t^n + 2\, \Sigma\, (a_t + 1)^n + \Sigma\, (b_t + 2)^n,$$
$$n = 1, 2, \ldots, l + 2.$$

Repeating this process m times, we obtain

$$\Sigma\, a_t^n + mC_1\, \Sigma\, (b_t+1)^n + {}_mC_2\, \Sigma\, (a_t+2)^n + {}_mC_3\, \Sigma\, (b_t+3)^n + \dots \dots$$
$$= \Sigma\, b_t^n + {}_mC_1\, \Sigma\, (a_t + 1)^n + {}_mC_2\, \Sigma\, (b_t + 2)^n + \dots (n = 1, \dots, l+m).$$
$$\dots \quad (3)$$

Now $1 + 4 = 2 + 3$. So

$$1^n + 4^n + (2 + 1)^n + (3 + 1)^n = 2^n + 3^n + (1 + 1)^n + (4 + 1)^n$$

i.e., $$1^n + 2.4^n = 2.2^n + 5^n. \qquad (n = 1, 2).$$

So $1^n + 2.4^n + 2.(2+1)^n + (5+1)^n = 2.2^n + 5^n + (1 + 1)^n + 2.(4+1)^n$

i.e., $$1^n + 2.3^n + 2.4^n + 6^n = 3.2^n + 3.5^n, \qquad (n = 1, 2, 3)$$

and so on.

If we start from $2+2 = 1+3$, we get the example given in I.

III. Let N (k) be the smallest N such that

$$\sum_{t=1}^{N} x_t^n = \sum_{t=1}^{N} y_t^n, \qquad (n = 1, 2, 3, \dots, k)$$

(where the x_t's are not identical with y_t's) has got infinity of primitive solutions. Then what can we say about N (k)?

$$N(1) = 2; \qquad N(2) = 3; \qquad N(5) = 6, \text{ etc.}$$

If $$\sum_{r=1}^{k} a_r^n = \sum_{r=1}^{k} b_r^n, \qquad (n = 1, 2, \dots, k),$$

then, from the theory of equations, it is easy to deduce that

$$a_r = b_r, \qquad r = 1, 2, \dots, k.^*$$

Hence if follows that $\quad N(k) \geqslant k + 1.$

Since $x + y = u + v$ has got infinity of primitive solutions, by putting $\quad a_1 = x,\ a_2 = y,\ b_1 = u,\ b_2 = v,\ l = 1,$

we get from (3) that $N(k) \leqslant 2^k$.

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* It follows that corresponding symmetric functions of a_r and b_r are equal. Hence a_r and b_r are the roots of the same equation of kth degree.

Thus we obtain $$k + 1 \leqslant \mathrm{N}(k) \leqslant 2^k.$$

But, if we start from $1 + 4 = 2 + 3$, then from (3) we get

$$1^n + 4^n + {}_mC_1(3^n + 4^n) + {}_mC_2(3^n + 6^n) + {}_mC_3(5^n + 6^n) + \ldots\ldots$$
$$= 2^n + 3^n + {}_mC_1(2^n + 5^n) + {}_mC_2(4^n + 5^n) + \ldots\ldots,$$

for $n = 1, 2, 3, \ldots, m$.

Let T be the number of terms on each side when we cancel out common terms. Then it can be easily verified that $\mathrm{T} < 4\ {}_mC_\mu$ where μ is the integral part of $m/2$.

So we get that

$$\sum_{t=1}^{r} a_t^n = \sum_{t=1}^{r} b_t^n, \qquad (n = 1, 2, \ldots, m) \quad \ldots \quad (4)$$

has got at least one solution when $r = \mathrm{T}$. Hence from the general remark in I, it follows that (4) has got infinity of primitive solutions when $r = 2\mathrm{T}$.

So $$\mathrm{N}(m) \leqslant 2\mathrm{T} = \mathrm{O}({}_mC_\mu).$$

Hence $$\mathrm{N}(k) = \mathrm{O}(2^k/\sqrt{k})$$

But I am not able to improve the result further.

GLEANINGS.

29. "Time was when a theorem could constitute a considerable contribution to Mathematical Science. But now theorems are turned out wholesale. A single treatise will contain hundreds of them. Nowadays methods alone can arrest attention strongly; and these are coming in such flocks that the next step will surely be to find *methods of discovering methods*. This can only come from the theory of the method of discovery. In order to cover every possibility, this could be founded on the general doctrine of methods of attaining purposes, in general; and this, in turn, should spring from the still more general doctrine of the nature of teleological action, in general."

C. S. PEIRCE.

30. There is no necessity for supposing that the process of thought as it takes place in the mind, is always cut up into distinct arguments.

C. S. PEIRCE.

TETRAHEDRAL POLES AND POLAR PLANES.

BY NATHAN ALTSHILLER COURT,
University of Oklahoma, U. S. A.

1. Given two skew lines a, b, and a point P, the line is drawn through P meeting a and b in the points A, B. The harmonic conjugate Q of P with respect to A, B, is said to be the "harmonic conjugate of P with respect to a, b."

2. *If a point describes a plane, the harmonic conjugate of this point with respect to two skew lines also describes a plane.*

Let C, D be the traces of the given lines a, b in the given plane, and let P be *any* point of this plane. The harmonic conjugate Q of P with respect to a, b lies in the plane CDQ which is harmonically separated from the given plane CDP by the two fixed planes CD-a, CD-b, hence the plane CDQ is fixed.

3. The plane CDQ may be called the "harmonic congugate" of the plane CDP with respect to the two given skew lines a, b. It is evident that the relation between the two planes CDP, CDQ is reciprocal.

4. *If a point describes a straight line, the harmonic conjugate of this point with respect to two skew lines also describes a straight line.*

Indeed, the harmonic conjugates Q, Q$'$, Q$''$, ... of the points P, P$'$, P$''$, ... of the given line u with respect to the two given skew lines a, b must lie in the harmonic conjugate plane, with respect to a, b of any plane passing through u, hence they lie on a line u'.

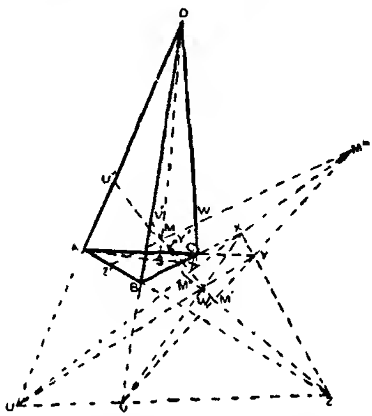
5. The line u' may be called the "harmonic conjugate" of u with respect to a, b.

It is clear that the relation between u and u' is reciprocal, and, furthermore, that the lines a, b are harmonic conjugates with respect to u, u'.

6. *If a line meets two opposite edges of a tetrahedron, its two harmonic conjugates with respect to the other two pairs of opposite edges coincide.*

Let U$'$, X$'$ be the traces of the given line U$'$X$'$ on the opposite edges DA, BC of the tetrahedron DABC. The harmonic conjugate

of U′X′ with respect to AB, CD is determined by the harmonic conjugates U, X of U′, X′ with respect to the pairs of points A, D; B, C. But the harmonic conjugate of the line U′, X′ with respect to the pair of lines AC, BD also contains the points U, X, as is readily seen by considering the transversals AU′D, BX′C, hence the proposition.



7. For the sake of brevity, we shall say that UX is the "harmonic conjugate of U′X′ with respect to the tetrahedron DABC." It is clear that the relation between U′X′ and UX is reciprocal.

8. The plane determined by the three harmonic conjugates of a given point with respect to the three pairs of opposite edges of a tetrahedron is referred to as the "harmonic plane," or the "polar plane," or the "tetrahedral polar plane" of the given point with respect to the tetrahedron.

9. *The harmonic conjugates, with respect to a tetrahedron, of the three lines passing through a given point and meeting the three pairs of opposite edges of the tetrahedron lie in the harmonic plane of the point with respect to the tetrahedron.*

Let d, e, f be three lines passing through the given point O and meeting the three pairs of opposite edges a, a'; b, b'; c, c', of the given tetrahedron (T). The harmonic conjugate d' of d with respect to (T) passes through the harmonic conjugates of O with respect to b, b'; c, c'. Hence d' lies in the harmonic plane of O with respect to (T). Similarly for e, f.

10. The point determined by the three harmonic conjugates of a given plane with respect to the three pairs of opposite edges of a tetrahedron is referred to as the "harmonic pole," or the "tetrahedral pole," or the "pole" of the given plane with respect to the tetrahedron.

11. *The harmonic conjugates, with respect to a tetrahedron, of three co-planar lines meeting, respectively, the three pairs of opposite edges of a tetrahedron, pass through the harmonic pole, with respect to the tetrahedron, of the plane containing the three given lines.*

Let p, q, r be the three given lines lying in the plane σ and meeting, respectively, the three pairs of opposite edges a, a'; b, b'; c, c' of the tetrahedron (T).

The harmonic conjugate p' of p with respect to (T) lies in the two harmonic conjugate planes of σ with respect to b, b'; c, c'. Hence p' passes through the pole of σ with respect to (T). Similarly for q and r.

12. *If μ is the polar plane of the point M with respect to a tetrahedron, then M is the pole of the plane μ with respect to this tetrahedron.*

Let u', v', w' be the three lines passing through M and meeting the three pairs of opposite edges of the tetrahedron (T). The harmonic conjugates u, v, w of u', v', w' with respect to (T) lie in the plane μ (9).

On the other hand the harmonic conjugates of u, v, w with respect to (T) pass through the pole of μ with respect to (T) (11). Now these harmonic conjugates coincide with u', v', w' for the relation between conjugate lines is reciprocal (7), hence the pole of μ with respect to (T) coincides with M.

13. If U′, X′; U, X are the traces of the lines u', u (12) on the edges DA, BC of (T) ≡ DABC, the two groups of points (DA, UU′), (BC, XX′) are harmonic (4). Similarly for the traces V′, Y′; V, Y of v', v on the edges DB, CA and again for the traces W′, Z′; W, Z of w', w on DC, AB. Hence the following table of groups of harmonic points

(A)	(DA, UU′)	(DB, VV′)	(DC, WW′)
	(BC, XX′)	(CA, YY′)	(AB, ZZ′)

14. *The trilinear pole, with respect to a face of a given tetrahedron DABC, of the line of intersection of this face with a given plane*

μ is collinear with the vertex opposite the face considered and the pole of μ with respect to DABC.

The pole M of μ lies on the lines U′X′, V′Y′, W′Z′ (12, 13), hence the trace of DM in ABC is the point S common to the lines AX′, BY′, CZ′. Now the trace of μ in the plane ABC contains the points X, Y, Z, hence S is the trilinear pole of XYZ.

15. *The trilinear polar, with respect to a face of a given tetrahedron DABC, of the trace in this face of the line joining the opposite vertex to a given point M lies in the polar plane of M with respect to DABC.*

The trace S of DM in ABC is common to the three lines AX′, BY′, CZ′ (14), hence the trilinear polar of S with respect to ABC is determined by the points X, Y, Z (13), and these three points lie in the polar plane μ of M with respect to DABC.

16. The harmonic conjugates M′, M″, M‴, of M (12) with respect to the pairs of points U′, X′; V′, Y′; W′, Z′ (13) determine the tetrahedral polar plane of M with respect to the tetrahedron DABC (8), so that we have (13)

$$\mu \equiv M'M''M''' \equiv UVWXYZ.$$

The points M″, M‴ are the traces on u = UX of the lines $v' \equiv$ V′Y′, $v \equiv$ VY. Now v', v are harmonic conjugates with respect to DA, BC, hence M″, M‴ are harmonically separated by U, X. Similarly for the lines $v \equiv$ VY, $w \equiv$ WZ We arrive thus at the following groups of harmonic points

$$\text{(B)} \qquad \begin{matrix} (MM', U'X'), & (MM'', V'Y'), & (MM''', W'Z') \\ (M''M''', UX), & (M'M''', VY), & (M'M'', WZ) \end{matrix}$$

17. If the vertices of a tetrahedron (P) are the harmonic poles of the opposite faces of (P) with respect to a second tetrahedron (Q), then (P) is said to be "self-polar," or "harmonic" with respect to (Q).

18. *A point and its three harmonic conjugates with respect to the three opposite edges of a given tetrahedron are four vertices of a tetrahedron harmonic to the given tetrahedron.*

The table (B) (16) shows that through M′ pass three lines M′U′X′, M′VY, M′WZ, and these meet the three pairs of opposite edges of DABC in the three pairs of points U′, X′; V, Y; W, Z; moreover

(B) shows that the harmonic conjugates of M' with respect to these three pairs of points are M, M''', M'' respectively. Consequently $MM''M'''^{1}$ is the harmonic plane of M' with respect to DABC. Similarly for the points M'', M'''^{1}. Thus $MM'M''M'''^{1}$ is harmonic with respect to DABC.

19. *If the tetrahedron (P) is self-polar with respect to the tetrahedron (Q), then (Q) is self-polar with respect to (P).*

Given the tetrahedron (P)=DABC and the point M (18) we constructed the tetrahedron (Q) = $M'M''M'''^{1}M$ self-polar with respect to (P). It follows from this construction that M is the vertex of one and only one such tetrahedron. Hence in order to prove the present proposition, it is sufficient to show that ABCD is harmonic with respect to $MM'M''M'''$.

From the harmonic group (DA, UU′) of table (A) (13) and the two harmonic groups (MM', $U'X'$), ($M''M'''$, UX) of table (B) it follows that the line DUU′ meets the opposite edges MM', $M''M'''^{1}$ of the tetrahedron $MM'M''M'''$ in the points U′, U and that A is the harmonic conjugate of D with respect to U′, U. Considering the lines DVV′, DWW′, we may show that the plane ABC is the polar plane of D with respect to the tetrahedron $MM'M''M'''$.

A similar reasoning may be applied to each of the points A, B, C. Hence the proposition.

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31. Women screamed and hundreds of penitents walked barefooted in the streets of the towns throughout Portugal last night in fear of a cataclysm when they saw the sky filled with millions of shooting stars. The unique display was due to the ***Earth's orbit passing through the orion nebula.***

Extract from an English newspaper referring to the meteor shower of 9th October 1933.

(*Italics ours*.—ED)

32. "In the curse of my law reading I constantly came upon the word "Demonstrate." I thought at first that I understood its meaning, but soon became satisfied that I did not. I consulted all the dictionaries and works of reference I could find * * At last I said, 'Lincoln, you can never make a lawyer if you do not understand what DEMONSTRATE means,' and I left my situation in Springfield, went home to my father's house and stayed there till I could give any proposition in the six books of Euclid at sight. I then found out what demonstrate means and went back to my law studies."

ABRAHAM LINCOLN.

NOTES AND DISCUSSIONS.

The Editor welcomes for publication under this heading, brief discussions of interesting problems, critical comments, and suggestions likely to be helpful in the class-room.

On a certain Theorem in Derangements.

In this note, I give a new method for determining in how many ways an ordered aggregate of n letters may be deranged so that no one out of r assigned letters shall occupy its original position.

Let $a_1, a_2, \ldots a_n$ be the n letters in order. There is no loss of generality in supposing that the r assigned letters are $a_1, a_2, \ldots a_r$. Let

$$\Sigma = a_1 + a_2 + \ldots + a_n$$

$$\sigma_1 = a_1 + a_2 + \ldots + a_r$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

σ_s = sum of all the s-ary combinations of the r letters $a_1, a_2 \ldots a_r$,

$$\omega = a_1 a_2 \ldots a_n.$$

In any derangement of letters fulfilling the conditions of the problem, the first place can be occupied by any one of the letters $a_2, a_3 \ldots a_n$.

That is, all the possibilities of filling the first place are found among the terms in $a_2 + a_3 + \ldots + a_n$ which is equal to $(\Sigma - a_1)$.

Similarly the independent possibilities for the 2nd, 3rd, ... rth places are given by $(\Sigma - a_2)$, $(\Sigma - a_3)$, ... $(\Sigma - a_r)$ and for each of the remaining $(n - r)$ places it is Σ.

Hence the totality of all derangements of the letters, fulfilling the conditions of the problem, is in (1, 1) correspondence with the various ways of forming the term $\omega \equiv a_1, a_2 \ldots a_n$ in the product P,

where $\quad P \equiv (\Sigma - a_1)(\Sigma - a_2) \ldots (\Sigma - a_r)\, \Sigma^{n-r}$

$$= \Sigma^n - \sigma_1 \Sigma^{n-1} + \ldots + (-)^s \sigma_s \Sigma^{n-s} + \ldots + (-)^r \sigma_r \Sigma^{n-r} \ldots \quad (1)$$

so that the required number is the co-efficient of ω in P,

Now σ_s is the sum of all the ${}_rC_s$ s-ary combinations of the r letters $a_1, a_2, \ldots a_r$, and each term gives ω once, when multiplied by the product of the remaining $(n-s)$ letters, and the co-efficient of this last product in Σ^{n-s} is clearly $(n-s)!$ Hence the co-efficient of ω in $\sigma_s . \Sigma^{n-s}$ is ${}_rC_s . (n-s)!$

Therefore the co-efficient of ω in P is

$$n! - {}_rC_1 (n-1)! + {}_rC_2 (n-2)! - \ldots + (-)^r (n-r)!$$

which is the number of derangements.

Taking $r = n$, we get the number of derangements of n letters in which no one letter occupies its original position, namely,

$$n! - {}_nC_1 (n-1)! + {}_nC_2 (n-2)! - \ldots + (-)^n . 1.$$

The theorem was first proved by Euler making use of a recurrence formula, and a proof on the same lines is given in Chrystal's *Algebra*, Part II.

R. S. VAIDISWARAN.

On Powers of Numbers.

Following the notation of a preceding paper,* I here prove

THEOREM 1. $(1) \overset{5}{=} (9)$ *i.o.*

THEOREM 2. $(2) \overset{5}{=} (8)$ *i.o.*

THEOREM 3. $(3) \overset{5}{=} (6)$ *i.o.*

We have†

$$(x+y+z)^5 = (y+z-x)^5 + (z+x-y)^5 + (x+y-z)^5 + 80xyz(x^2+y^2+z^2) \ldots (1)$$

Putting $y = 25n^5$, $z = 50n^5$ and $x = 4k^5$

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* K. Subba Rao "On Powers of Numbers," *The Math. Student*, Vol. II, No. 1 (1934), pp. 6–10.

† This identity is taken from S. Sastry's paper to appear in *Jour. Lond. Math. Soc.*

in (1), we have

$$(4k^5 + 75n^5)^5 = (75n^5 - 4k^5)^5 + (4k^5 + 25n^5)^5 + (4k^5 - 25n^5)^5 + 2(20n^2k^3)^5 + 4(50n^4k)^5 \quad \dots \quad (2)$$

If in (2), $\qquad 25n^5 < 4k^5 < 75n^5,$

we have $\qquad (1) \overset{5}{=} (9) \quad i.o.$

On the other hand, if in (2), $\qquad 4k^5 < 25n^5$

we have $\qquad (2) \overset{5}{=} (8) \quad i.o.$

Hence Theorems 1 and 2 are proved.

Again, we have

$$\begin{aligned}(10t^5 + 1)^5 - (10t^5 - 1)^5 &= (10t^4)^5 + 2000(t^2)^5 + 2 \\ &= (10t^4)^5 + (5t^2)^5 - 1125(t^2)^5 + 1^5 + 1^5 \quad \dots \quad (3)\end{aligned}$$

Now $\qquad 1125 \times 5^5 = 20^5 + 12^5 + 9^5 + 6^5 - 2^5.$

Substituting this in (3) and rearranging, we have

$$\begin{aligned}(50t^5 + 5)^5 &+ (20t^2)^5 + (12t^2)^5 + (9t^2)^5 + (6t^2)^5 \\ &= (50t^5 - 5)^5 + (50t^4)^5 + (25t^2)^5 + (2t^2)^5 + 5^5 + 5^5\end{aligned}$$

which proves Theorem 3.

K. SUBBA RAO.

On the tests for the divisibility of an integer by another.

Tests for the divisibility of an integer by 3, 4, 9, 11 are well-known. In this note, I discuss the rationale of these and other similar tests.

Since we can have only n different remainders when integers are divided by n, at least two of the numbers in the set $1, 10, 10^2, \dots, 10^n$ should leave the same remainders. Let the first such pair be 10^a and 10^{a+k}. Then $10^{a+k} - 10^a$, that is, $10^a(10^k - 1)$ is a multiple of n. Now $10^{a+mk+t} - 10^{a+t} = 10^t . 10^a . (10^{mk} - 1)$ and $10^{mk} - 1$ is a multiple of $10^k - 1$. So $10^{a+mk+t} - 10^{a+t}$ is a multiple of n. Hence, if $r_1, r_2, r_3 \dots$ are the remainders when $1, 10, 10^2, 10^3 \dots$ are divided by n, it follows that $r_{a+mk+t} = r_{a+t}$. That is,

$$r_{a+t} = r_{a+t+k} = r_{a+t+2k} = \dots, \quad (t = 0, 1, \dots, k-1).$$

Consequently, the remainders, when 1, 10, 10^2 ... are divided by n, form the periodic sequence

$$1,\ r_1,\ r_2,\ \ldots\ r_{a-1},\ \dot{r}_a,\ r_{a+1},\ \ldots,\ \dot{r}_{a+k-1} \qquad \ldots \quad (1)$$

where $1, \ldots, r_{a-1}$, is the non-periodic part and $r_a, r_{a+1}, \ldots r_{a-k-1}$ are repeated in the same order.

Let $R(x/y)$ be the remainder when x is divided by y, and let $N = d_m d_{m-1} \ldots d_1 d_0$ in the usual decimal notations. Then

$$\begin{aligned} R(N/n) &= R\{(d_0 + 10d_1 + \ldots + 10^m d_m)/n\} \\ &= R\{(d_0 + r_1 d_1 + \ldots + r_m d_m)/n\} \\ &= R\{[d_0 + r_1 d_1 + \ldots + r_{a-1} d_{a-1} \\ &\quad + \sum_{t=0}^{k-1} r_{a+t}(d_{a+t} + d_{a+t+k} + d_{a+t+2k} + \ldots)]/n\} \end{aligned}$$

from (1) and (2).

(1) $n = 3$, $1 = r_0 = r_1 = r_2 = \ldots \ldots$
$R(N/3) = R\{(d_0 + d_1 + \ldots + d_m)/3\}$.

(2) Similarly for 9.

(3) $n = 8$, $r_0 = 1$, $r_1 = 2$, $r_2 = 4$, $r_3 = r_4 = \ldots \ldots = 0$.
So $R(N/8) = R\{(d_0 + 2d_1 + 4d_2)/8\} = R\{(d_2 d_1 d_0)/8\}$.

(4) $n = 12$ $\quad r_0 = 1$, $r_1 = 10$, $r_2 = r_3 = r_4 = \ldots \ldots = 4$.
So $R(N/12) = R\{[d_0 + 10d_1 + 4(d_2 + d_3 + \ldots + d_m)]/12\}$.

(5) $n = 11$, $\quad r_0 = r_2 = r_4, \ldots = 1$; $r_1 = r_3 = \ldots \ldots = 10$.

So $R(N/11) = R\{[d_0 + d_2 + d_4 + \ldots + 10(d_1 + d_3 + d_5 + \ldots)]/11\}$.

So if N is divisible by 11, then

$0 = R(N/11) = R\{[d_0 + d_2 + d_4 + \ldots + (11-1)(d_1 + d_3 + \ldots +)]/11\}$
$= R\{[(d_0 + d_2 + \ldots) - (d_1 + d_3 + \ldots)]/11\}$.

Hence $(d_0 + d_2 + \ldots) - (d_1 + d_3 + \ldots \ldots)$ is divisible by 11.

(6) $n = 125$, $r_0 = 1$, $r_1 = 10$, $r_2 = 100$, $r_3 = r_4 = \ldots = 10$.
So $R\{N/125\} = R\{(d_2 d_1 d_0)/125\}$, etc.

A. M. DESHPANDE.

An operational solution of $\frac{dy}{dx} + Py = Q$ where P and Q are functons of x only.

1. If P and Q are functions of x, an operational solution of $\frac{dx}{dy} + Py = Q$ or $(D + P)\,y = Q$, is given by

$$y = \frac{1}{D + P}\,Q \qquad \text{... (1)}$$

or
$$y = \frac{1}{D}\left\{1 - \left(\frac{P}{D}\right) + \left(\frac{P}{D}\right)^2 - + \dots\right\} \qquad \text{... (2)}$$

where
$$\frac{P}{D}Q = PD^{-1}Q = P\int Q\,dx.$$

$$\left(\frac{P}{D}\right)^r Q = \left(\frac{P}{D}\right)\left(\frac{P}{D}\right)^{r-1} Q \qquad (r \geqslant 2).$$

Example: If
$$\frac{dy}{dx} + xy = x$$

then
$$y = \frac{1}{D}\left\{1 - \frac{x}{D} + \left(\frac{x}{D}\right)^2 - + \dots\right\} x = \frac{x^2}{2} - \frac{x^4}{8} + - \dots$$
is a solution.

It is easy to show that (3) *is, under certain convergence conditions, actually a solution of the given differential equation.*

From (2) we also get

$$y = \frac{1}{P}\left\{1 - \frac{D}{P} + \left(\frac{D}{P^2}\right) - + \dots\right\} Q \qquad \text{... (3)}$$

where
$$\frac{D}{P}Q = D\left(\frac{Q}{P}\right) = \frac{d}{dx}\left(\frac{Q}{P}\right)$$

$$\left(\frac{D}{P}\right)^r Q = \frac{D}{P}\cdot\left(\frac{D}{P}\right)^{r-1} Q \qquad (r \geqslant 2).$$

Example: If $\frac{dy}{dx} + xy = x$, then $y = 1$ is a solution.

It is important to observe that *in* (2) *we do not, in general, get a solution if we interpret*

$$\frac{P}{D}Q \quad \textit{as} \quad \frac{1}{D}(PQ).$$

Similarly *in* (3) *we cannot interpret* $\frac{D}{P}\,Q$ *as meaning*

$$\frac{1}{P}\,.\,D\,(Q) = \frac{1}{P}\,\frac{dQ}{dx}.$$

2. An operational solution of

$$\frac{d^2y}{dx^2} + P\,\frac{dy}{dx} + Qy = R$$

where P, Q, R are functions of x only, is given by

$$y = D^{-2}\left(1 + \frac{P}{D} + \frac{Q}{D^2}\right)^{-1} R$$

$$= D^{-2}\Bigg[1 - \left(\frac{P}{D} + \frac{Q}{D^2}\right)$$

$$+ \left\{ \left(\frac{P}{D}\right)^2 + \left(\frac{Q}{D^2}\right)^2 + \left(\frac{P}{D}\right)\left(\frac{Q}{D^2}\right) + \left(\frac{Q}{D^2}\right)\left(\frac{P}{D}\right) \right\}$$

$$-\Bigg\{ \left(\frac{P}{D}\right)^3 + \left(\frac{Q}{D^2}\right)^3 + \left(\frac{Q}{D^2}\right)^2\left(\frac{P}{D}\right) + \left(\frac{Q}{D^2}\right)\left(\frac{P}{D}\right)\left(\frac{Q}{D^2}\right)$$

$$+ \left(\frac{P}{D}\right)\left(\frac{Q}{D^2}\right)^2 + \left(\frac{P}{D}\right)^2\left(\frac{Q}{D^2}\right) + \left(\frac{P}{D}\right)\left(\frac{Q}{D^2}\right)\left(\frac{P}{D}\right)$$

$$+ \left(\frac{Q}{D^2}\right)\left(\frac{P}{D}\right)^2 \Bigg\}$$

$$+ - \ldots \Bigg] R$$

where $\quad \frac{P}{D}\,R = PD^{-1}R, \quad \frac{Q}{D^2}\,R = QD^{-2}R$, etc., etc.

The symbols used are not commutative and any other interpretation $\left[e.g., \frac{P}{D}R = \frac{1}{D}(PR)\right]$ *would, in general, lead to false results.*

It is evident that we can proceed similarly with equations of higher order than the second.

I. CHOWLA.

SOLUTIONS TO QUESTIONS.

Question 1617.

(R. VAIDYANATHASWAMY) :—The θ normal at a point P of a conic is defined to be the line obtained by rotating the tangent at P through an angle θ in the positive direction about P. A θ-normal which passes through the centre of the conic is called a θ-diameter.

If the sides of a triangle inscribed in a conic are respectively parallel to the θ_1, θ_2 and θ_3 diameters of the conic, show that the θ normals to the conic at its vertices are concurrent, θ being given by

$$\cot\theta = \cot\theta_1 + \cot\theta_2 + \cot\theta_3.$$

Solution by S. Narayanan.

Let the eccentric angles of the vertices of the triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be α, β, γ. Since the θ-normals at the vertices are concurrent, $2ab\cot\theta = c^2\,\Sigma\sin(\beta+\gamma)$ ... (1) (*Vide* Casey's *Analy. Geometry*, page 539, Question 116).

The side given by

$$\frac{x}{a}\cos\frac{\beta+\gamma}{2}+\frac{y}{b}\sin\frac{\beta+\gamma}{2}=\cos\frac{\beta-\gamma}{2}$$

is parallel to the θ_1 diameter of the conic. The θ_1 diameter at any point 'ϕ' of the ellipse is $y=\frac{bx}{a}\tan\phi$ where

$$\cot\theta_1 = -\frac{a^2-b^2}{ab}\sin\phi\cos\phi$$

By the condition of parallelism, $\tan\phi = -\cot\frac{\beta+\gamma}{2}$.

$$\therefore\quad \cot\theta_1 = \frac{a^2-b^2}{2ab}\sin(\beta+\gamma).$$

$$\Sigma\cot\theta_1 = \frac{a^2-b^2}{2ab}\,\Sigma\sin(\beta+\gamma) = \cot\theta \text{ by (1)}.$$

Question 1627.

(R. VAIDYANATHASWAMY) :—A surface S supposed rigid is moved by an infinitesimal rotation about an axis L into the position S′. If S′ touches S at O, prove that L must lie in one of the principal normal sections at O and pass through the centre of curvature of the other principal normal section.

Solution by P. Jagannathan.

If A is any point on the axis L and if by an infinitesimal rotation about L, any point O be carried into O′, then AO = AO′ and OO′ is perpendicular to AO.

Now let P be the point in which the axis L meets the tangent plane at O. By the infinitesimal rotation, T_0, the tangent plane at O, is carried into T_0', the tangent plane at O′ to S′. In the limit PO is the intersection of T_0 and T_0' and OO′ is perpendicular to PO so that PO and OO′ besides being conjugate directions on the surface at O are also principal directions.

Let G be the point in which the plane π containing OO′ perpendicular to PO meets L. Then GO = GO′ and OO′ is perpendicular to GO. Therefore GO and GO′ are the normals at O and O′ and G the centre of curvature in the plane π. The plane DOG is the other principal plane.

Hence the required result.

Question 1638.

(P. V. SUBBA RAO) :—Let a and b be two integers prime to each other, and N any integer. Let q_1 and r_1 be the quotient and the remainder when Na is divided by b; q_2 and r_2 the quotient and remainder when q_1a is divided by b, and so on, the quotient at each stage being multiplied by a and divided by b to get the next quotient and remainder. If this be done k times, show that

(i) If $N \leqslant b^k$, the sequence of remainders $(r_1, r_2, r_3, \dots, r_k)$ are different for different numbers;

(ii) If $A_1, A_2, \dots, A_k$ are values of N for which the sequence of remainders are $(1, 0, 0, 0 \dots)$, $(0, 1, 0, 0, \dots)$, $\dots$, $(0, 0, \dots, 1)$

respectively, then the number $N \leqslant b^k$ for which the remainders are $(r_1, r_2, \ldots, r_k)$ is congruent to $r_1 A_1 + r_2 A_2 + \ldots + r_k A_k$, (mod b^k).

(iii) Show how to determine the numbers $A_1, A_2, \ldots, A_k$ in any particular case, say for $a = 4$, $b = 3$, $k = 4$,

Solution by Hansraj Gupta.

(i) If possible let N and M be two different numbers $\leqslant b^k$ having the same sequences of remainders, *viz.*, $(r_1, r_2, r_3, \ldots r_k)$.

Then we must have

$$aN = q_1 b + r_1 \quad , \quad aM = q'_1 b + r_1,$$
$$q_1 a = q_2 b + r_2 \quad , \quad q'_1 a = q'_2 b + r_2,$$
$$\ldots \ldots \ldots \ldots \quad , \quad \ldots \ldots \ldots \ldots$$
$$q_{k-1} a = q_k b + r_k \quad , \quad q'_{k-1} a = q'_k b + r_k.$$

$$\therefore \quad a^k (N - M) = a^{k-1} b (q_1 - q'_1), \quad = a^{k-2} b^2 (q_2 - q'_2) = \ldots$$
$$= b^k (q_k - q'_k) \equiv 0. \quad (\text{mod } b^k).$$

Since a, b are relatively prime, $N - M \equiv 0$ (mod b^k).

This contradicts the supposition, since N and M are different and not greater than b^k. Hence the proposition.

(ii) We have

$$aA_m = q_{m,1} b; \quad aq_{m,1} = q_{m,2} b,$$
$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots$$
$$aq_{m,m-2} = q_{m,m-1} b, \quad aq_{m,m-1} = q_{m,m} b + 1, \quad aq_{m,m} = q_{m,m+1} b,$$
$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$
$$aq_{m,k-1} = q_{m,k} b. \qquad \ldots \text{(A)}$$

$m = 1, 2, 3, \ldots, k.$

$$\therefore \quad aN' = a \sum_{m=1}^{k} r_m A_m = \sum_{m=2}^{k} r_m (aA_m) + r_1 (aA_1) = b \sum_{m=1}^{k} r_m q_{m,1} + r_1$$

$$a \sum_{m=1}^{k} r_m q_{m,1} = b \sum_{m=1}^{k} r_m q_{m,2} + r_2,$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$a \sum_{m=1}^{k} r_m q_{m,k-1} = b \sum_{m=1}^{k} r_m q_{m,k} + r_k$$

where $$N' = \sum_{m=1}^{k} r_m A_m \qquad \ldots \quad (1)$$

Therefore the number $N \leqslant b^k$ for which the sequence of remainders is $(r_1, r_2, r_3, \ldots, r_k)$ is congruent to $\sum_{m=1}^{k} r_m A_m \pmod{b^k}$. ... (2)

(iii) From the set of equations (A) given above, we have

$$a^{m-1} A_m = b^{m-1} q_{m,\, m-1}; \qquad a^{k-m} q_{m,\, m} = b^{k-m} q_{m,\, m};$$

$\therefore$ $$q_{m,\, m-1} = a^{m-1} y, \quad \text{where } y \text{ is an integer,}$$

and $$q_{m,m} = b^{k-m} x, \quad \text{where } x \text{ is an integer.}$$

Hence $$a^m y = b^{k-m} (bx) + 1 = b^{k-m+1} x + 1 \qquad \ldots \quad (3)$$

This is an indeterminate equation of the first degree and gives y.

Now $A_m = b^{m-1} y$ gives A_m. Or A_m may be obtained directly from the equation

$$a^m A_m = b^k x + b^{m-1}. \qquad \ldots \quad (4)$$

In the particular case when $a = 4$, $b = 3$, $k = 4$, we have

$$4A_1 = 81\, x_1 + 1. \quad \therefore A_1 = 81\, S_1 + 61.$$

Similarly $A_2 = 81S_2 + 66$, from the equation $16A_2 = 81x_2 + 3$;

$A_3 = 81S_3 + 9$, " $64A_3 = 81x_3 + 9$;

and $A_4 = 81S_4 + 27$, " $256A_4 = 81x_4 + 27$.

Question 1652.

(A. A. KRISHNASWAMI AYYANGAR) :—The tangents at two points P, Q of a conic meet in T. Show that the circle passing through T and touching PQ at P or Q cuts orthogonally the director circle of the conic. Hence deduce Gaskin's theorem of which the above is a special case : " The circumcircle of a triangle self-conjugate with respect to a conic cuts the director circle orthogonally."

Solution by A. Ranganatha Rao.

It is well-known that the director circles of a system of conics touching four lines are co-axal. There are three point-pairs which are the degenerate members of the system and the director circles corresponding to these are the circles described on their joins as diameters. Now, let the four lines coalesce two by two so that we have two lines TP, TQ, where TP is one coalesced pair of lines, P being the point of ultimate intersection and TQ is the other pair of coalesced lines, Q being the point of intimate intersection. The conics now become a system of conics Σ that touch TP, TQ at P, Q respectively. Of the three point-pairs of the system, only one remains distinct, namely the point-pair P, Q, while the remaining four points constituting the other two pairs are absorbed at T. The director circles of Σ form, therefore, a co-axal system S to which belong the circle described on PQ as diameter and the point-circle T (a limiting point).

Now, let p and q be the circles which pass through T and touch PQ at P and Q respectively. It is obvious that the circles p and q are orthogonal to the circle described on PQ as diameter and since they also pass through T, they must be orthogonal to every circle of the system S. This proves the required result.

Again, the circles p and q define a co-axal system S$'$, every member of which is orthogonal to every member of the system S. Also the circles of S$'$ intersect PQ in pairs of points belonging to an involution, of which the double points are P and Q. If one pair of points in the involution be A, B, then A, B are conjugate points with respect to every conic of Σ, and hence TAB is a self-polar triangle of all those conics. Thus the circle TAB is orthogonal to the director circles of the conics of Σ, which proves Gaskin's theorem.

Also solved by S. Narayanan, M. K. Hariharan and analytically by S. P. Ranganathachar and A. L. Shaikh.

Question 1667.

(A. A. KRISHNASWAMI AYYANGAR):—Prove that

$$\int_0^x e^{-\frac{1}{2}x^2}\,dx > \sqrt{\frac{\pi}{2}}\,(1-e^{-x^2}), \qquad (x>0).$$

Solution by A. L. Shaikh.

The above inequality is equivalent to

$$\int_0^a e^{-x^2}\,dx > \frac{\sqrt{\pi}}{2}\,(1-e^{-\frac{1}{2}a^2}). \qquad \dots \quad (1)$$

For
$$\int_0^a e^{-\frac{1}{2}x^2}\,dx = \sqrt{2}\int_0^{a/\sqrt{2}} e^{-x^2}\,dx,$$

so that we have to show that

$$\sqrt{2}\int_0^{a\sqrt{2}} e^{-x^2}\,dx > \sqrt{\frac{\pi}{2}}\,(1-e^{-a^2})$$

or replacing $a\sqrt{2}$ by a on both sides, we have to prove (1).

This is easily obtained by a method used in Gibson's *Calculus* p. 340.

Let
$$I = \int_0^a e^{-x^2}\,dx = \int_0^a e^{-y^2}\,dy$$

Then
$$I^2 = \int_0^a e^{-x^2}\,dx \times \int_0^a e^{-y^2}\,dy$$
$$= \iint e^{-(x^2+y^2)}\,dx\,dy$$

taken over the square OABC $(0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant a)$ of side a. Since the integrand is positive in this region, the above integral is greater than the same taken over the quadrant of the circle $r = \frac{a}{\sqrt{2}}$; $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

$$\therefore \qquad I^2 > \iint e^{-(x^2+y^2)}\,dx\,dy \text{ taken over the quadrant.}$$

Changing into polar co-ordinates

$$I^2 > \int_0^{a/\sqrt{2}} e^{-r^2}\,r\,dr \times \int_0^{\pi/2} d\theta = \frac{\pi}{4}\,(1-e^{-a^2/2})$$

$$\therefore \qquad I > \frac{\sqrt{\pi}}{2}\left(1-e^{-a^2/2}\right)^{\frac{1}{2}} > \frac{\sqrt{\pi}}{2}\left(1-e^{-a/2^2}\right)$$

since $0 < 1-e^{-a^2/2} < 1$.

Question 1676.

(A. A. KRISHNASWAMI AYYANGAR) :—Show that there is a unique solution to the letters A, E, G, S and the missing digits, which all make the following multiplication true. No two different letters represent the same digit.

$$
\begin{array}{cccccccc}
 & & & E & * & * & * & S \\
 & & & & & A & G & E \\
\hline
 & & * & * & * & E & G & * \\
 & & * & * & G & * & E & \\
G & E & * & * & * & A & & \\
\hline
G & A & * & * & E & * & * & *
\end{array}
$$

Solution by K. Subba Rao.

From the data given, we have

$$E \geqslant 3 \quad \dots (a); \qquad G \times E < 10. \qquad \dots (b)$$

Hence $G \leqslant 2$. Also

$$G \times S \equiv E \pmod{10}. \qquad \dots (c)$$

G cannot be 1, for then $S = E$. So $G = 2$.

From (a) and (b), we have $E = 3$ or 4. $\dots (d)$

Now (c) becomes $2S \equiv E \pmod{10}$. $\dots (c_1)$

From (d) and (c_1), we have $E = 4$, $S = 7$

$A \times S \equiv A \pmod{10}$, *i.e.*, $(7A \equiv A \bmod 10)$.

So $A = 5$. Now let the multiplicand be EXYZS.

From the above, we must have $X = 8$ or 9. $4Z + 2 \equiv 2 \pmod{10}$. So $Z = 0$ or 5.

If $Z = 0$, $4Y \equiv 4 \pmod{10}$, *i.e.*, $Y = 1$ or 6.

If $Z = 5$, $4Y + 2 \equiv 4 \pmod{10}$, *i.e.*, $Y = 3$ or 8.

Finally, since we have E $(=4)$ in the thousands place in the product, we should have $X = 9$, $Y = 6$, $Z = 0$.

Hence the solution is unique.

Also solved by Hansraj Gupta, N. Maitra, N. K. Narasimhamurti, T. K. Raghavachari, S P. Ranganathachar and A. Ranganatha Rao.

ANNOUNCEMENTS AND NEWS.

Readers are invited to contribute to the value of this section by sending suitable news items of interest.

The following gentlemen have been elected members of the Society —

N. P. Subramania Aiyar, M.A., Maharajah's College, Ernakulam.

K. G. Panikkar, M.A. Hons.), Maharajah's College, Ernakulam

J. Veeravenkiah, B A., B. Ed, Veeresalingam High School, Rajahmundry.

The Committee of the Indian Mathematical Society has been reconstituted for the years 1934–36 The Committee consists of

President H. G Gharpurey, I.C.S, (Retd.), 344, Shanwarpet, Poona.

Treasurer Prof. L N. Subramaniam, M A., Madras Christian College, Madras

Librarian Prof. V. B. Naik M.A., Fergusson College, Poona.

Other Members

Dr. R. Vaidyanathaswamy, M.A., D.Sc., F.R.S.E.

Principal G. S. Mahajani, M.A., Ph.D.

Prof. A. C Banerji, M.A , M Sc., F.R A.S.

Prof Mukand Lal, M.A.

Principal T. K. Venkataraman, M.A.

Prof. Arunachala Sastri, M.A.

Prof. A Narasinga Rao, M.A., L.T.

Dr. Ram Behari Lal, M.A., Ph.D.

Prof. S. B. Belekar, M A.

The Committee has passed a resolution recording its appreciation of the services rendered to the Society by the retiring President and Members of the Committee.

The Second Karnataka Astronomical Conference was held at Mysore between the 13th and 16th of October 1934 under the gracious patronage of Their Highnesses the Maharaja and the Yuvaraja of Mysore and the Presidentship of Sir Vepa Ramesam, *Kt.*, Judge, High Court of Judicature, Madras. The Yuvaraja of Mysore, in opening the Conference, urged the need for a reform of the Indian Almanac at the hands of scholars well-versed in Mathematics and Astronomy and skilled in observation and possessed of a critical mind. The President pointed ont the usefulness of an Indian Observatory and appealed to the enlightened Ruler and the Dewan of Mysore to crown their constructive work by the creation of an Observatory for the study of celestial phenomena.

More than thirty papers were read in four different languages (English, Samskrit Tamil and Kannada) relating to various controversal points in practical Hindu Astronomy, such as the exact longitude of the first point of Mesha, the maximum and minimum duration of a Tithi, etc. The programme inclnded also three public lectnres on astronomical subjects, illustrated by lantern slides, an exhibition of astronomical charts, models, instruments and calculating devices, and Spectroscopic and Telescopic demonstrations arranged with the co-operation of the Physics and Mathematics Departments of the Intermediate College, Mysore.

The Conference passed a resolution urging the need for an expert Board of Studies in Hindu Astrronomy to work in consultation with the Mysore University Board of Studies in Mathematics to carry out the aims of this Conference.

Mr. A. S. Besicovitch of Cambridge and the Rev. Alfred Young, Rector of Birdbrook, Essex. have been elected Fellows of the Royal Society of London.

Dr. Adolf Fraenkel, formerly of Kiel, has been appointed Professor of Mathematics at the Hebrew University, Jerusalem.

Prof. Henri Lebesgue, of the University of Paris, has been elected a foreign Member of the Royal Society of London.

Dr. B M. Mehrotra, M.A. (Agra), Ph D. (Liverpool), has been appointed Assistant Prof. of Mathematics at the Benares Hindu University.

Our readers will be glad to learn that the collected papers of Prof. G. A. Miller are to be published by the University of Illinois The first volume of about 500 pages is expected to be published by the middle of 1935 and will contain some original historical articles in addition to Prof. Miller's earliest contributions to Group Theory. The whole work is expected to extend to six volumes.

The following publications are now being received in exchange for those of the Indian Mathematical Society.

Annali di Mathematica pura ed Applicata, Bologna.

Sitzungsberichte der Preussiche Akedemie der Wissenschaften (Math. Phys. Klasse), Berlin.

Science Progress, A quarterly review of Scientific thought, work and affairs, published by Edward Arnold & Co., London.

H. S. Hall, author of well-known text-books in Mathematics and formerly Head of the Military and Engineering side of Clifton College, died May 3. 1934 at the age of 85.

BOOKS RECEIVED FOR REVIEW.

G. F. C. Searle: EXPERIMENTAL PHYSICS, a selection of Experiments; Cambridge University Press, 1934, pp. 363. Price 16s.

R. W. Gurney: ELEMENTARY QUANTUM MECHANICS. Cambridge University Press, 1934, pp. 160. Price 8s. 6*d*.

Sri Niwas Asthana: The Trisection of an angle leading to divide an angle into any number of equal parts, pp. 14. Price 1-8-0. Published by the Author.

QUESTIONS FOR SOLUTION.

Proposers of Questions are requested to send their own Solutions along with their Questions.

1683. (A. RANGANATHA RAO) :—D, E, F are the points of contact of the in-circle of a triangle ABC with the sides BC, CA, AB, and PP′ is any diameter of the in-circle. Prove that if any one of the following properties is true, so are the others.

(i) The pedal circle of P with respect to ABC passes through P′;

(ii) The polar axis of P with respect to ABC is perpendicular to PP′;

(iii) P′, D′, E′, F′ being the diametrically opposite points to P, D, E, F; D′P, E′P, F′P meet BC, CA, AB respectively in three collinear points;

(iv) The triad (DEF) is apolar to the triad consisting of P′ and the point P repeated twice, *i.e.*, to (P′PP).

1684. (V. RAMASWAMI AIYAR) :—From a variable point O on the θ-normal at a point P of a conic, three other θ-normals are drawn to the conic, their feet being L, M, N. Prove that the circum-circle of LMN, as O varies, forms a co-axal system whose common points are Q, the point on the conic diametrically opposite to P, and M_1 the foot of the θ-perpendicular drawn from the centre of the conic on the tangent at Q.

[*Note.*—If the θ-normal at P cut the tangent at Q in Q_1, M_1 is the mid-point of QQ_1.]

1685. (V. RAMASWAMI AIYAR) :—In the foregoing problem prove further that the sides of the triangle LMN, as O varies, envelope a parabola whose focus is M_1.

1686. (A. A. KRISHNASWAMI AYYANGAR) :—If $\int_a^b x^n f(x) = 0$ for all integral $n \geqslant 0$, is $f(x)$ identically zero?

1687. (V. RAMASWAMI AIYAR AND A. NARASINGA RAO) :—Prove that perpendicular tangents to any parabola inscribed in an equilateral triangle are also principal axes of a conic inscribed in its medial triangle.

1688. (A. A KRISHNASWAMI AYYANGAR) :—Prove the following properties of the circles of curvature of a parabola :

(*a*) No two circles of curvature can have external contact, nor can they lie, each outside the other ;

(*b*) No circle of curvature can touch internally more than two other circles of curvature ;

(*c*) Each circle of curvature encloses two infinite sets of circles of curvature ; it is also enclosed in two other infinite sets of such circles.

1689. (V. RAMASWAMI AIYAR AND A. NARASINGA RAO) :—ABCD is an ortho-centric tetrad of points and l is any straight line. Parabolas are drawn inscribed in each of the triangles ABC, BCD, CDA, DAB and touching l. Show that all the parabolas touch another line l', and that l and l' are the principal axes of conics inscribed in the medial triangle of ABCD.

1690. (V. RAMASWAMI AIYAR) :—A conic passing through the in- and ex- centres of a triangle ABC has one asymptote parallel to BC. Show that that asymptote passes through A and the other asymptote passes through A$'$, the point diametrically opposite to A on the circum-circle of ABC. If DEF be the medial triangle of ABC, show that the conic passes through D, and the tangent to it at D is the median AD. Show that the conic also passes through the foot of the perpendicular from A on EF ; and that the tangent to it at the point passes through the foot of the perpendicular from A$'$ on BC

1691. (A. B. LAL) :—A particle is projected along the inside of a smooth surface of revolution, whose axis is vertical, with a velocity v at a point P of the surface such that the direction of motion makes an angle ψ with the meridian curve through that point. Show that the initial radius of curvature of the path of the particle is given by

$$\frac{1}{\rho'^2} = \frac{1}{\rho^2} + \frac{g^2 \sin^2 \psi \sin^2 \phi}{v^4}$$

where ρ is the normal curvature at P in the direction of motion, and ϕ is the angle which the normal to the surface at P makes with the vertical.

(This is a generalisation of a problem in § 210 of Besant and Ramsay's *Dynamics*).

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ON EXPRESSING THE EQUATIONS OF ANY CONGRUENCE OF CURVES IN THE HAMILTONIAN FORM.

By K. NAGABHUSHANAM,
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1. Let the equations of a given congruence of curves in an n dimensional manifold be

$$x^r = c_r \quad (r = 1, 2, \ldots n-1); \quad x^n = \text{anything} \quad \ldots \quad (A)$$

where $c_1, c_2, \ldots c_{n-1}$ are arbitrary constants.

We inquire if the equations (A) can be expressed in the Hamiltonian form

$$\left.\begin{aligned} dq^\alpha - \frac{\partial H}{\partial p_\alpha}\, dt &= 0 \\ -\, dp_\alpha - \frac{\partial H}{\partial q^\alpha}\, dt &= 0 \end{aligned}\right\} (\alpha = 1, 2, \ldots r) \ldots \quad (B)$$

2. Suppose that (A) can be put in the form (B) by a change of variables. On changing the variables in (B) to $x^1, x^2, \ldots x^n$ the equations (B) take the form*

$$a_{ik}\, dx^k = 0, \qquad (i = 1, 2, \ldots n) \qquad \ldots \quad (C)$$

where $a_{ik} = \dfrac{\partial X_i}{\partial x^k} - \dfrac{\partial X_k}{\partial x^i}$, and the repeated index stands for summation from 1 to n. The problem then reduces to finding the covariant vector (X_i) such that the Pfaff's First System of equations (C) of the form

* See E. T. Whittaker: *Analytical Dynamics*, 3rd ed., p. 308.

$X_i\, dx^i$ is identical with (A). The co-ordinate system $(x^1, x^2 \ldots x^n)$ is used throughout the discussion. We shall treat the two cases when n is odd and even separately.

2. (a) $$n = 2r + 1.$$

Suppose that it is possible to write (A) in the form (C). Substituting $dx^1 = dx^2 = \ldots dx^{n-1} = 0$ in (C), we have

$$a_{in}\, dx^n = 0, \qquad (i = 1, 2, \ldots n).$$

As $dx^n \neq 0$ along the curves of the congruence,

$$a_{in} = 0, \qquad (i = 1, 2, \ldots n). \qquad \ldots \quad \text{(D)}$$

Also the number of independent equations of the congruence is $n - 1$ ($= 2r$), so that the rank of the matrix $\| a_{ik} \|$ is exactly $2r$. Whenever (X_i) satisfies the two conditions

(i) $a_{in} = 0, \qquad (i = 1, 2 \ldots n).$

and (ii) the rank of $\| a_{ik} \|$ is $2r$, the equations (C) yield (A) as the very conditions of consistency.

(b) $n = 2r + 2$

The set of equations got by leaving out the first equation in (A) can be brought under the form (C), as seen in (a). In this case, the $2r$ independent equations in (C) must be supplemented by the equation $x^1 =$ constant

In both cases the $2r$ equations in the form (C) can be written* in the form (B) by a suitable change of the variables to $q^1, \ldots q^r, p_1, \ldots p_r$. We may state the result as follows :

The equations (A) are equivalent to a system of Hamiltonian equations when n is odd and to a set of Hamiltonian equations together with an equation of the type $\phi =$ constant (ϕ being an integral of the congruence) when n is even.

---

* In the next section we shall show how to construct the form $X_i dx_i$. When it is expressed as $\sum_{\alpha=1}^{r} p_\alpha dq^\alpha - H\, dt$, the two forms (B) and (C) become identical.

3. Lastly, we shall determine the form of (X_i). From (D) we have

$$\frac{\partial X_j}{\partial x^{2r+1}} = \frac{\partial X_{2r+1}}{\partial x^j}, \qquad (j = 1, 2 \ldots 2r).$$

If we write

$$X_j = \frac{\partial \psi}{\partial x_j} + f_j (x^1, x^2, \ldots x^{2r})$$

and

$$X_{2r+1} = \frac{\partial \psi}{\partial x^{2r+1}} + f_{2r+1} (x^{2r+1})$$

where ψ is an arbitrary function of all the variables, then

$$X_i dx^i = \frac{\partial \psi}{\partial x^i} dx^i + f_i dx^i,$$

i.e.,

$$(X_i) = \text{grad } \psi + (f_i) \qquad \ldots \quad (E)$$

Since f_{2r+1} is a function of x^{2r+1} only, we may write

$$(f_i) = \text{grad } \phi + (f_1, f_2, \ldots f_{2r}, 0)$$

where $\phi = \int f_{2r+1} dx^{2r+1}$.

Now (E) becomes

$$(X_i) = \text{grad } (\psi + \phi) + (f_1, f_2, \ldots f_{2r}, 0) \qquad \ldots \quad (F)$$

The equation contains in itself the condition (D). The second condition that the rank of $\| a_{ik} \|$ is $2r$ is equivalent to the statement that the rank or class* of the form $\omega_{2r} = \sum_{j=1}^{2r} f_j dx^j$ is $2r$. Also any of the functions $f_1, f_2, \ldots f_{2r}$ is either a constant or an integral of the congruence.

I hereby thank Dr. R. Vaithianathaswamy for his kind help.

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* The rank cannot be greater than $2r$, for ω_{2r} is expressed in $2r$ variables. It cannot be less than $2r$ which is the rank of $\| a_{ik} \|$.

THEOREMS CONCERNING TWO HOMOGRAPHIC TETRADS OF POINTS ON A CONIC.*

By T. R. Raghava Sastri,

Loyola College, Madras.

Two homographic ranges on a conic S are uniquely determined by two triads of corresponding points (A, B, C) and (A′B′C′) on it. If D, D′ are a pair of corresponding points, the corresponding cross joins like AD′, A′D meet on the homographic axis and the joins of corresponding points like AA′ envelop a conic having double contact with S at the double points.

The purpose of this paper is to draw attention to certain results relating to two such tetrads

Theorem. If (ABCD) aad (A′B′C′D′) are two homographic *tetrads of points on a conic S, the six meets of pairs of corres*ponding joins like AB, A′B′ together with the six meets of pairs of non-corresponding joins like AB, C′D′ lie on a conic R passing through the pole P with respect to S of the homographic axis of the two tetrads ... (A)

Let BC, B′C′ meet in X and AD, A′D′ in X′ with Y, Y′, Z, Z′ denoting similar meets. Let BC and A′D′ meet in L, B′C′ and AD in L′, with M, M′, N, N′ denoting similar meets. Also let the homographic axis meet the conic S in H and H′.

We start with the fundamental fact that

PX and YZ, PY and ZX and PZ and XY are conjugate pairs for S ... (1.1)

This follows from the Pascal hexagon CABC′A′B′ wherein the Pascal line YZ passes through the meet of BC′ and B′C which is evidently the pole of PX for S. Similarly it can be proved that the other pairs are also conjugate for the conic.

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* I am indebted to Mr. V. Ramaswamy Aiyar for helpful suggestions, one of them being the enunciation of the theorem (A) in the present form.

Hence the polar of P (*i.e.*, the homographic axis) cuts the opposite pairs of sides of the quadrangle PXYZ in involution with H, H′ as double points. Hence also the pencil of conics through PXYZ will determine the same involution on HH′.

By taking the two corresponding triads B, C, D and B′, C′, D′, it similarly follows that the pencil of conics through PXY′Z′ determines the same involution on HH′. Evidently therefore

the conic PXYY′Z must pass through Z′ as well ; and by parity of reasoning through X′ also ... (1·2)

Again, from (1·1), PX, YZ as well as PX, Y′Z′ are conjugate pairs for the conic S. Therefore YZ and Y′Z′ must intersect in the pole of PX which is necessarily on the polar of P, *i.e.*, the homographic axis. Similarly ZX and Z′X′, XY and X′Y′ must intersect on this line. But from (1.2) the six points lie on a conic R. Hence from the hexagon XYZX′Y′Z′,

the line HH′ must contain the meet of ZX′ and Z′X ; whence XX′ and ZZ′, similarly YY′ also, pass through the pole of HH′ with respect to the conic R ... (1.3)

If HH′ meets R in I and I′, it follows therefore that each pair X, X′ ; Y, Y′ ; Z, Z′ divide I, I′ harmonically. As P is a point on R, P (XX′II′) = − 1. Hence their poles for S form a harmonic range on HH′. But the pole of PX is the meet of BC′ with HH′ and the pole of PX′ is the meet of AD′ with HH′ Also the poles of PI and PI′ are I′ and I respectively. The pencil of conics through ABC′D′, of which S is one, cuts HH′ in involution ; but as (HH′, II′) and the above range are harmonic, therefore I, I′ are the double points of this involution. Hence AB and C′D′ cut II′ dividing it harmonically. Therefore N (ZZ′, II′) = − 1 ; whence N is also a point on the conic. By similar reasoning it can be shown that L, M, L′, M′, N′ are also on R ... (1 4)

Hence Theorem (A) is completely proved.

Cor. The tretrad (ABCD) is also homographic with each of the tetrads (B′A′D′C′), (C′D′A′B′) or (D′C′B′A′). Hence by application of theorem A, the same conic R passes through the poles P_1, P_2, P_3 of these three homographic axes also ... (1.5)

Cor. If the poles of AA′, BB′, CC′, DD′ for S be denoted by α, β, γ, δ it follows that the conic R is the pole-locus of the corresponding homographic axis for all conics passing through $\alpha\beta\gamma\delta$. Similarly it is seen that R is the pole locus of the corresponding homographic axis for all conics passing through the poles of joins of corresponding points of ABCD and any of the tetrads homographic with it mentioned in the above corollary.*

Cor. When the two tetrads are in involution, the conic R degenerates into a line pair one of which is the axis of involution. Hence the six meets like (AB, C′D) and the centre of the involution are collinear. It may be noticed that P_1, P_2, P_3 mentioned in (1.5) coalesce with P.

2. It remains to notice certain special cases. To begin with, we may note that as the two triads ABC, A′B′C′ determine uniquely the homographic ranges, the theorem A) can be restated in the useful form

If two triangles ABC and A′B′C′ are inscribed in a conic S and D′ be any point on the conic, then the meets of D′A′, D′B′ and D′C′ with BC, CA, AB respectively and the meets of the corresponding sides of the two triangles lie on a conic passing through the pole of the Pascal line of the hexagon AB′CA′BC′ ... (2.1)

Further, if any straight line meets the sides BC, CA, AB of a triangle inscribed in a conic S, in L, M, N and if D′ be any point on the conic, then D′L, D′M, D′N meet the conic again in A′B′C′ such that the triangles ABC and A′B′C′ are in perspective. Moreover the centre of perspective lies on the straight line. The converse of this theorem is also true ... (2.2)

The theorem (2.2) gives other particular cases. When S is a circle, the pedal line and Poncelet's theorems are seen to be deductions.

As another example we may consider this theorem given in Lachlan's *Pure Geometry*:—O is a point on the circle circumscribing a triangle ABC. Points A′, B′, C′ are taken on the sides such that AA′, BB′, CC′ subtend right angles at O. Show that A′, B′, C′ are collinear and the line of collinearity passes through the centre of the circle. The truth is at once seen from the converse of the theorem (2.2).

* This gives another method of proof of theorem (A).

3. I have considered some well-known theorems which are deducible from (2.2). The following appear to be both new and interesting:—

If A, B, C, P are four concyclic points and lines PL, PM, PN are drawn making the same angles θ in the same sense with the sides of the triangle ABC, meeting the sides in L, M, N and any conic through ABCP in A′B′C′, then the triangles ABC and A′B′C′ will be always in perspective. Also the centre of perspective traces the line LMN as the conic is varied. For, since L, M, N are collinear, therefore the two triangles ABC and A′B′C′ are in perspective from (2.2). The centre of perspective always lies on this line LMN.

Question 1662 of this journal is another deduction from (2.2).

4. Lastly there are two particular cases of theorem (A) which are worthy of notice.*

When the points I and I′ (where R cuts the homographic axis) are projected to circular points, the interesting theorem given in Q. 1433 is obtained. The circle R in this case becomes the nine-point circle of the ortho-centric tetrad $\alpha\beta\gamma\delta$ mentioned in (1.6) (4.1)

Secondly when H, H′ are projected into circular points, we get a theorem concerning two directly similar quadrangles inscribed in a circle, the R conic in this case becoming a rectangular hyperbola ..(4.2)

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* Mr. V Ramaswamy Aiyar drew my attention to Q. 1433 of which theorem (A) is the generalisation and also to the deduction (4.2).

ON A CERTAIN PROBLEM IN DETERMINING THE EXTREMA OF A FUNCTION OF SEVERAL VARIABLES.

By A. NARASINGA RAO,
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1. *The Problem Stated : Existence Theorem.*

In Goursat's classical treatise on Mathematical Analysis,* one finds an analytical discussion of the problem of determining the position of a point P the sum of whose distances from the vertices of a triangle is a minimum. The object of this note is to state the problem in mechanics equivalent to the generalised problem and by this means discriminate between the proper solutions when P is distinct from any of the vertices, and the improper solutions when it coincides with one of them.

The generalisation we shall consider may be stated thus :

Given any number of points $A_1, A_2 \ldots A_n$, not necessarily co-planar and a single valued and continuous function $f(r)$, it is required to determine the position of a point P such that

$$\phi(P) = a_1 f(r_1) + a_2 f(r_2) \ldots + a_n f(r_n)$$

shall be a minimum, $a_1, a_2, \ldots$ being positive constants and $r_1, r_2, \ldots$ the distances $PA_1, PA_2, \ldots$ etc. ... (1.1)

We shall confine ourselves to the case when $f(r)$ increases monotonically and becomes infinite with r, and $f(r) > 0$ for $r > 0$. The totality of values $\phi(P)$ is bounded below since $\phi(P) \geqslant 0$, and has hence a lower limit. This limit must be actually attained at one or more points P_0 since $\phi(P)$ is a continuous function of position. As the number of vertices A_r is finite, there exists a convex polygon (if the points are co-planar) or a convex polyhedron whose vertices are all included in A_r and such that every A_r is either an interior point or is situated on a face or edge or at a vertex. Calling this polyhedron (or, polygon) Δ, it is clear that P_0, a position of P for which $\phi(P)$ is a minimum,

* Goursat : *Mathematical Analysis*, Translated by Hedrick, Vol I, Ch III, Art. 62.

cannot be outside Δ for then all the points A_r lie on the same side of a plane π through P_0. An infinitesimal movement of P towards this side would diminish all the distances PA_r and hence also the value ϕ (P). The same argument shows also that P_0 cannot be on a face or edge of Δ, for an infinitesimal movement of δs towards the interior would diminish all the distances PA_r by an amount of the same order as δs, except in the case of the A-points lying in the face or edge whose distances are increased by an infinitesimal of order $(\delta s)^2$.

We thus have two types of solutions according as P_0 is an interior point of Δ other than the A-points (proper solution) or is coincident with a vertex of Δ (improper solution). ... (1.2)

2. *The Equivalent Problem in Mechanics.**

Consider n gravitational masses a_1, a_2 ... situated at A_1, A_2, ... and let the law of attraction between two masses at distance r be $mm' f'(r)$. We have then a field of force in which the potential energy of a unit mass at P is ϕ (P) + const. Hence a minimum value of ϕ (P) corresponds to a position of stable equilibrium at P. Our existence theorem proves that there is at least one position of stable equilibrium.

The simplest case is the one discussed by Goursat in which $f(r) = r$, and the force along PA_r is accordingly a_r. We thus have the following experimental method of determining P_0: Take a smooth table with small holes at A_1, A_2,...A_n through each of which pass strings carrying hanging weights a_1, a_2, ... a_n. Let the other ends of the strings (those on the table) be knotted together at P. The positions of stable equilibrium give the points for which $\Sigma\, a_k r_k$ is a relative minimum (*i.e.*, less than at all other points in its immediate neighbourhood).†

If $n = 3$ and $a_1 = a_2 = a_3$ (so that the weights are equal) we have Fermat's problem discussed by Goursat. It is clear from statical

* The advantages of the introduction of the energy concept in the discussion of this problem was pointed out to me by Mr. V. Ramaswami Aiyar.

† If l be the length of each of the strings, the depth of the centre of gravity of the system below the table is $\sum\limits_k a_k (l - r_k)$ which is a maximum when $\Sigma\, a_k r_k$ is a minimum.

considerations that the strings will be equally included so that P_0 (the Fermat Point*) is one at which each of the sides subtends an angle of 120°.†

If one of the angles of the triangle, say A_1, exceeds 120°, there is then no such point and we have an improper solution with P_0 at A_1.

More generally, if $n = 3$ and a_1, a_2, a_3 are unequal, P_0 is a point (if such exists) such that P_0A_1, P_0A_2, P_0A_3 are parallel to the sides of a triangle (of forces) whose lengths are a_1, a_2, a_3; so that the angles subtended at P_0 are the supplements of the angles of the triangle $a_1a_2a_3$.‡

For a convex quadrilateral $n = 4$ and let $a_1 = a_2 = a_3 = a_4$. We have a rhombus for the force polygon so that the strings are collinear in pairs. Hence P_0 is the intersection of the diagonals and there is no other proper solution.

In the case of a triangle, we saw that improper solutions arise owing to there being no point P_0 at which the sides subtend the proper angles. Their occurrence may also be explained statically as due to one of the masses, say a_r, being too large to allow of equilibrium except when P is at A_r. Thus suppose a_r is greater than the magnitude of the vector sum of forces a_1 along $A_r A_1$, a_2 along $A_r A_2$, ... a_{r-1} along $A_r A_{r-1}$, a_{r+1} along $A_r A_{r+1}$ ... and a_n along $A_r A_n$.

It is then clear that if P is displaced from A_r, the weight a_r will drag the knot P back to A_r. Thus with three points there will be an improper solution with P_0 at A_1 when

$$a_1^2 > a_2^2 + a_3^2 - 2a_2 a_3 \cos A_2\hat{A}_1A_3,$$

that is, when $\angle A_2A_1A_3$ is less than the angle opposite a_1 in a triangle with sides a_1, a_2, a_3. If $a_1 = a_2 = a_3$ we get Goursat's criterion for an improper Fermat Point, namely, one of the angles of $A_1A_2A_3$ exceeding 120°.

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* *Vide* Edwards : *Differential Calculus*, 1929, p 430, Ex. 18

† For four points A, B, C, D, not in a plane, the Fermat point P_0 is such that any pair of opposite edges subtend equal angles at P_0 (Wolstenholme : Problem No. 2229).

‡ *Vide* S. Subramanyan : *Annamalai University Journal*, Vol. III, p. 60.

For n points, the required criterion for an improper solution at A_1 is that a_1 is greater than the resultant of forces a_2, a_3 ... ... along A_1A_2, A_1A_3, ... *

The plane problem with n points A_r is also that of minimising the total distance of travel of populations $a_1, a_2, \ldots$ at A_1 A_2, ... to a common place P_0, under the somewhat artificial assumption that each person travels straight from his home to P_0, and has attracted some attention for $n = 3$ in the *Annamalai University Journal*.† The statistical criteria obtained in this paragraph apply equally to the geometrical problem with the A-points distributed in space.

3. When $f(r) = r^2$, matters are considerably simpler. The force system consists of forces $a_k r_k$ along PA_k whose resultant is a force $(a_1 + \ldots a_n)$ PG where G is the centroid of masses $a_1, a_2 \ldots$ at $A_1, A_2 \ldots$ Hence

If $f(r) = r^2$, there is only one position of P which minimises ϕ (P), namely, the centre of mean position of A_1, A_2, ... for multiples $a_1, a_2, \ldots$

Other forms of $f(r)$ may be discussed similarly, but the results are not easily expressible in terms of well-known geometrical entities.

4. Lastly, it remains to point out that while proper solutions correspond to positions P_0 for which $\phi(P) - \phi(P_0)$ is an infinitesimal of a higher order than the distance PP_0 and hence satisfy the relations

$$\frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial \phi}{\partial y} = 0, \quad \frac{\partial \phi}{\partial z} = 0$$

where $\phi(P) = \phi(x, y, z)$; this is not true of improper solutions. Thus in a problem with an improper solution at A_1, let P be a point very near A_1 on the locus

$$a_2 r_2 + a_3 r_3 + \ldots a_n r_n = a_2 A_1 A_2 + a_3 A_1 A_3 \ldots + a_n A_1A_n.$$

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* An obvious case when this happens is when one of the weights is greater than the sum of all the others; *Vide* S. Subramanyan: *Annamalai University Journal*, Vol. III, p. 61.

† S. Subramanyan, *loc cit*,

G. V. Krishnaswami: A note on the Centre of Population, *Annamalai University Journal*, Vol. III, pp. 278-282.

G. V. Krishnaswami: *Annamalai University Journal*, Vol. IV, No. 1,

which obviously passes through A_1. Then $\delta\phi(P) = \phi(P) - \phi(A_1) = a_1 r_1$ approximately and this is of the same order as PA_1. Hence the first partial derivatives do not vanish at A_1.

For the plane problem, $\phi(P) = \phi(x, y)$ is a function of position given by

$$\phi(x, y) = \sum_1^n a_r \sqrt{(x-\xi_r)^2 + (y-\eta_r)^2} \qquad (r = 1, 2, \dots n)$$

where ξ_r, η_r are the co-ordinates of A_r and x, y those of P. The surface $z = \phi(x, y)$ is found to have conical points corresponding to the points $A_r(\xi_r, \eta_r)$. The point on the surface nearest to the xy plane may be either an ordinary point at which the tangent is parallel to the xy plane (proper solutions) for which

$$\frac{\partial\phi}{\partial x} = 0, \qquad \frac{\partial\phi}{\partial y} = 0,$$

or one of the conical points, in which case the partial derivates do not vanish.

ON CERTAIN CONGRUENCES.

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1. Let $\lambda(p^\alpha) = \phi(p^\alpha)$, p being an odd prime;

$\lambda(2^\alpha) = \phi(2^\alpha)$, $\alpha \leqslant 2$; $\lambda(2^\alpha) = \frac{1}{2}\phi(2^\alpha)$, $\alpha > 2$;

and $\lambda(m)$ = the L. C. M. of $\lambda(p_1^{\alpha_1})$, $\lambda(p_2^{\alpha_2})$, ... $\lambda(p_\mu^{\alpha_\mu})$ where $m = p_1^{\alpha_1} p_2^{\alpha_2} \dots \dots p_\mu^{\alpha_\mu}$ (p's being different primes).

We know that if a be prime to m, $a^{\lambda(m)} \equiv 1 \pmod m$.

If d be any divisor of $\phi(p^\alpha)$ where p is a prime, the number of ntegers less than and prime to p^α belonging to exponent d mod p^α is $\phi(d)$.

If l be the L. C. M. of $d_1, d_2, d_3, \dots d_\mu$, where d_1 is a divisor of $\phi(p_1^{\alpha_1})$, d_2 of $\phi(p_2^{\alpha_2})$, ... d_μ of $\phi(p_\mu^{\alpha_\mu})$, the number of integers

less than and prime to m belonging to exponent d mod m is

$$\Sigma\,[\phi(d_1).\phi(d_2)\,\ldots\,\phi(d_\mu)].$$

For, if $a_1, a_2 \ldots a_{\phi(d_1)}$ be the integers less than and prime to $p_1^{\alpha_1}$ belonging to exponent d_1 mod $p_1^{\alpha_1}$, they satisfy the congruence $x^{d_1} \equiv 1$ mod $p_1^{\alpha_1}$. And if $b_1, b_2 \ldots b_{\phi(d_2)}$ be such integers for $p_2^{\alpha_2}$, they satisfy the congruence $x^{d_2} \equiv 1$ mod $p_2^{\alpha_2}$.

Now, if y and z be so chosen that $a_r + p_1^{\alpha_1}\,y = b_\rho + p_2^{\alpha_2}\,z$ where a_r, b_ρ are numbers of the above sets, the number in either member of this equation will be a common root of the above congruences. Hence if l' be the L. C. M. of d_1 and d_2, the congruence $x^{l'} \equiv 1$ mod $p_1^{\alpha_1}p_2^{\alpha_2}$ has $\phi(d_1).\phi(d_2)$ roots less than and prime to $p_1^{\alpha_1}p_2^{\alpha_2}$ belonging to exponent l' mod $p_1^{\alpha_1}p_2^{\alpha_2}$-roots arising from the divisors d_1 and d_2. Similar remarks hold for the divisors d'_1 and d'_2 of $\phi(p_1^{\alpha_1})$ and $\phi(p_2^{\alpha_2})$ having l' as their L. C. M. Thus the number of integers prime to and less than $p_1^{\alpha_1}p_2^{\alpha_2}$ belonging to exponent l' mod $p_1^{\alpha_1}p_2^{\alpha_2}$ is $\Sigma\ \phi(d_1).\phi(d_2)$. This can be easily extended.

It follows that if $l = \lambda(m)$, the number of the primitive λ-roots of m is $N = \Sigma\ [\phi(d_1).\phi(d_2)\,\ldots\,\phi(d_\mu)]$ ... (A)

and the number of the primitive λ-roots of m corresponding to the L. C. M. $\lambda(m)$ of $\lambda(p_1^{\alpha_1})$, $\lambda(p_2^{\alpha_2})$, ... $\lambda(p^{\alpha_\mu})$ is

$$N' = \phi\,[\phi(p_1^{\alpha_1})].\phi\,[\phi(p_2^{\alpha_2})]\,\ldots\,\phi\,[\phi(p_\mu^{\alpha_\mu})] \ldots \text{(B)}$$

The primitive λ-roots of m in (B) may be called proper primitive λ-roots and others improper.

The primitive λ-roots of m can be divided into $\{N/\phi\,[\lambda(m)]\}$ sets, and those of any set can be derived from any root of that set. For example, let $m = 65$. $\lambda(m) = 12$. The divisors of $\phi(5)$ are $d_1 = 1$, $d_2 = 2$, $d_3 = 4$; and divisors of $\phi(13)$ are $\delta_1 = 1$, $\delta_2 = 2$, $\delta_3 = 3$, $\delta_4 = 4$, $\delta_5 = 6$, $\delta_6 = 12$, The L. C. M. of d_3 and δ_6; d_3 and δ_5; d_3 and δ_3; d_2 and δ_6; d_1 and δ_6 is $12 = \lambda(m)$. The number of the primitive λ-roots of 65 is

$$\phi(d_3).\phi(\delta_6) + \phi(d_3).\phi(\delta_3) + \phi(d_2).\phi(\delta_6) + \phi(d_1).\phi\delta_6)$$
$$= 2.4 + 2.2 + 2.2 + 1.4 + 1.4 = 24.$$

They are	(1)	2, 32, 63, 33	(corresponding to d_3 and δ_6)
	(2)	7, 37, 58, 28	(„ „)
	(3)	17, 62, 43, 23	(„ d_3 and δ_5)
	(4)	42, 22, 3, 48	(„ d_3 and δ_3)
	(5)	54, 19, 59, 24	(„ d_2 and δ_0)
	(6)	41, 6. 46, 11	(„ d_1 and δ_6)

They are easily obtained by solving certain indeterminate equations of the first degree. Thus

2 belongs to d_3 $(=4)$ mod 5, hence 2 is a root of $x^4 \equiv 1 \bmod 5$.

4 belongs to δ_5 $(=6)$ mod 13, hence 4 is a root of $x^6 \equiv 1 \bmod 13$.

Choose y and z so that $2 + 5y = 4 + 13z$. The number in either member of this equation is a common root of the above congruences. Hence 17 is a root of $x^{12} \equiv 1 \bmod 65$, *i.e.*, it is a primitive λ-root of 65.

Similarly the primitive roots of $2p^\alpha$ where p is an odd prime can be deduced from those of 2 and p^α.

2. Since $a^{\lambda(m)} \equiv 1 \bmod m$ where a is prime to and less than m, the congruence $ax \equiv -1 \bmod m$ has a unique solution $x = -a^{\lambda(m)-1}$ $= -\mu m + a'$, that is, $x = a'$ where a' is less than and prime to m. Hence integers prime to and less than m can be combined in pairs, such that

$$aa' \equiv -1 \bmod m \qquad \ldots \text{(C)}$$

unless $a' = a$, that is, unless $x^2 + 1 \equiv 0 \bmod m$. Now every odd prime divisor of $x^2 + 1$ where x is a positive integer is of the form $4n + 1$; and $x^2 + 1 \not\equiv 0 \bmod 4$. It follows that the congruence $x^2 + 1 \equiv 0 \bmod m$ has no solution when m is a prime of the form $4n - 1$ or has a prime factor of this form or $m = 2^\mu$ $(\mu > 1)$ or has a factor of this form. Thus if P denote the product of numbers less than and prime to m, then having due regard to the number of pairs of these numbers, we have the following :—

(i) $P \equiv -1 \bmod p$ (ii) $P \equiv -1 \bmod 4$

(iii) $P \equiv -1 \bmod p^\alpha$ (iv) $P \equiv -1 \bmod 2p^\alpha$

(v) $P \equiv -1 \bmod 2^\mu$ $(\mu > 2)$ (vi) $P \equiv 1 \bmod m$

where m has a factor of the form p^{α} or 2^{μ} $(\mu > 1)$ p being a prime of the form $4n - 1$ in all cases.

3. Since the congruence $x^2 + 1 \equiv 0 \bmod m$ cannot have more than two roots when m is a prime, it follows that if it has a root a, then it must have two roots a and $m - a$. It is easy to show that when m is a prime of the form $4n+1$, the congruence $x^2+1 \equiv 0 \bmod m$ has always two roots, one even and the other odd. If $4n$ is of the form r^2 then r and $m - r$ are the solutions.

If g be a primitive root of a prime $4n + 1$, then

$$g^{2n} - 1 \not\equiv 0 \bmod (4n + 1).$$

But since $g^{4n} - 1 \equiv 0 \bmod (4n + 1)$, it follows that

$$g^{2n} + 1 \equiv 0 \bmod (4n + 1) \qquad \dots \quad \text{(D)}$$

Hence one root of the congruence $x^2 + 1 \equiv 0 \bmod (4n + 1)$ is g^n which is congruent to a mod $(4n + 1)$ where $a < 4n + 1$. The other root is $(4n + 1) - a$.

Now 1 and $4n - 1$ are the least and greatest numbers less than and prime to $4n$. The numbers prime to and less than $4n$ can therefore be equally divided into two sets, one containing numbers of the form $1 + 4\lambda$, and the other containing numbers of the form $4n - 1 - 4\mu$.

Thus the primitive roots of $4n + 1$ falling in the first and second sets are of the form $g^{4\lambda+1}$ and $g^{4n-1-4\mu}$ where g is any primitive root of $4n + 1$. Since

$$(g^n)^{4\lambda+1} - (g^n)^{4\lambda'+1} = g^{(4\lambda'+1)n}\,(g^{4(\lambda-\lambda')n} - 1) \equiv 0 \bmod (4n+1),$$

and $$(g^n)^{4n-1-4\mu} - (g^n)^{4n-1-4\mu'} \equiv 0 \bmod (4n + 1),$$

the two roots are unique.

If γ and γ' be the roots of $x^2 + 1 \equiv 0 \bmod (4n + 1)$, then

$$\gamma\gamma' \equiv 1 \bmod (4n + 1). \qquad \dots \quad \text{(E)}$$

It follows from (C) that

(vii) $\qquad 4n\,! \equiv -1 \bmod (4n + 1)$.

If x be an even root, $x^2 + 1 = my$, $m > y + 1$.

$\therefore \qquad (m - x)^2 + 1 = 2mz \qquad m > z \geqslant 1$

where m is a prime $4n + 1$ and y, z are primes or products of primes of the same form.

Hence $$2x - y = m - 2z \qquad \dots \quad \text{(F)}$$

In simple cases, a few trials give x and hence g.

4. If g be a primitive root of p^α where p is a prime of the form $4n + 1$, then $g^{\phi(p^\alpha)} - 1 \equiv 0 \bmod p^\alpha$.

$\therefore \quad g^{\frac{1}{2}\phi(p^\alpha)} - 1 \not\equiv 0 \bmod p^\alpha$ and hence $g^{\frac{1}{2}\phi(p^\alpha)} + 1 \equiv 0 \bmod p^\alpha \dots$ (G)

It follows that $g^{\frac{1}{2}\phi(p^\alpha)}$, that is, $g^{np^{\alpha-1}}$ is a root of the congruence $x^2 + 1 \equiv 0 \mod p^\alpha$. Hence $p^\alpha - a$ is the other root where $a \equiv g^{np^{\alpha-1}} \bmod p^\alpha$ and $a < p^\alpha$. The numbers less than and prime to $4np^{\alpha-1}$ are of the form $1 + 4\lambda$ and $4np^{\alpha-1} - 1 - 4\mu$, and therefore the primitive roots of p^α can be divided into two sets as before.

Thus $x^2 + 1 \equiv 0 \bmod p^\alpha$ has two roots unique. If γ and γ' be the roots $$\gamma\gamma' \equiv 1 \bmod p^\alpha. \qquad \dots \quad \text{(H)}$$

Similar remarks hold for $x^2 + 1 \equiv 0 \bmod 2p^\alpha$, where p is a prime of the form $4n + 1$.

It follows from (C) and (H) that if P be the product of the numbers prime to and less than p^α (or $2p^\alpha$), then

(viii) $$P \equiv -1 \bmod p^\alpha \text{ (or mod } 2p^\alpha).$$

5. Consider the congruence $x^2 + 1 \equiv 0 \mod m$ where $m = 2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \dots p_\mu^{\alpha_\mu}$ (the p's being different odd primes).

If $\alpha \geqslant 2$ or if any odd prime be of the form $4n - 1$, it has no root. It follows from (C) that if P be the product of numbers less than and prime to m in this case, then

(ix) $$P \equiv 1 \bmod m.$$

If $\alpha = 0$ or 1 and each p be of the form $4n + 1$, and g be a proper primitive λ-root of m, then $g^{\frac{1}{2}\lambda(m)} - 1 \not\equiv 0 \bmod m$.

But $g^{\lambda(m)} - 1 \equiv 0 \bmod m$. $\therefore \; g^{\frac{1}{2}\lambda(m)} + 1 \equiv 0 \bmod m \;\dots$ (I)

$\therefore \quad g^{\frac{1}{2}\lambda(m)}$ is a root of $x^2 + 1 \equiv 0 \bmod m$. $m - a$ is another root where $a \equiv g^{\frac{1}{2}\lambda(m)} \bmod m$, a being prime to and less than m.

If γ, γ' be the roots, $\gamma\gamma' \equiv 1 \bmod m$ ... (J)

The number of proper primitive λ-roots of m which satisfy the congruence

$$x^{\frac{1}{2}\lambda(m)} + 1 \equiv 0 \bmod m \text{ is } N' = \phi\,[\phi\,(p_1^{\alpha_1})] \ldots \phi\,[\phi\, p_\mu^{\alpha_\mu})].$$

It may be pointed out that the improper primitive λ-roots of m do not satisfy this congruence, for they are derived from the congruences of degrees lower (at least one being of lower degree) than

$$\phi\,(p_1^{\alpha_1}),\ \phi\,(p_2^{\alpha_2}),\ \ldots\ \phi\,(p_\mu^{\alpha_\mu}).$$

The proper primitive λ-roots fall in pairs such that the numbers in each pair raised to the power $\frac{1}{2}\,\lambda(m)$ leave the same remainder when divided by m.

Hence $\frac{1}{2}$ N$'$ is the number of roots of $x^2 + 1 \equiv 0 \bmod m$, and these roots also fall in pairs such that the product of the numbers in each pair is congruent to 1 mod m.

Since N$'/4$ (the number of such pairs) is always even, it follows from (C) that if P be the product of numbers prime to and less than m,

(x) $$P \equiv 1 \bmod m.$$

NOTE. γ, γ' are, in general, other than the primitive roots of p^α (or $2p^\alpha$) or other than the primitive λ-roots of m such that

$$\gamma\gamma' \equiv 1 \bmod p^\alpha \text{ (or } 2p^\alpha) \text{ or } \bmod m.$$

It may be noted that the primitive roots or primitive λ-roots fall in pairs in two ways such that $gg' \equiv -1$ and $gg_{,} \equiv 1$.

6. Let $a_1, a_2, a_3 \ldots a_n$ be the numbers less than and prime to p^α (or $2p^\alpha$). Then it can be proved that

$$\left.\begin{aligned} x^{\phi(p^\alpha)} - 1 &\equiv (x - a_1)(x - a_2) \ldots (x - a_n) \bmod p^\alpha \\ x^{\phi(2p^\alpha)} - 1 &\equiv (x - a_1)(x - a_2) \ldots (x - a_n) \bmod 2p^\alpha \end{aligned}\right\} \ldots \text{(K)}$$

where p is an odd prime.

7. The theorem due to Gauss which was partially proved in my note 'On an extension of Wilson's Theorem' in *J. I. M. S.*, Vol. XIX, No. 2, has been proved above, but the following proof appears to be interesting.

(*a*) Let $m = p^{\mu}$ where p is a prime not less than 3 and $\mu > 1$, and $\phi(p^{\mu}) = n$. The integers prime to and less than m are

$$rp + 1,\ rp + 2,\ rp + 3, \ldots rp + p - 1 \qquad \ldots \quad (1)$$

where r varies from 0 to $p^{\mu-1} - 1$.

The numbers in (1) are a complete set of roots of the congruence $x^{n} - 1 \equiv 0 \bmod m$. If γ be a primitive root of m, then the numbers

$$\gamma,\ \gamma^2,\ \gamma^3, \ldots \gamma^n \qquad \ldots \quad (2)$$

are also all the incongruent roots of the congruence.

The numbers in (1) are therefore congruent in some order to the numbers in (2).

Denoting the numbers in (1) by $a_1, a_2, \ldots a_n$, we have (say)

$$a_1 \equiv \gamma \bmod m;\ a_2 \equiv \gamma^2 \bmod m;\ \ldots \ldots\ a_n \equiv \gamma^n \bmod m.$$

$$\therefore\quad a_1 a_2 \ldots a_n \equiv P \equiv \gamma^{\frac{n}{2}(n+1)} \equiv (\gamma^n)^{\frac{n}{2}} . (\gamma^n)^{\frac{1}{2}} \bmod m$$

where P is the product of numbers in (1); Hence $P - (1)^{\frac{1}{2}} \equiv 0 \bmod m$,

i.e., $\quad P + 1 \equiv 0$ or $P - 1 \equiv 0 \bmod m$.

Now the only term in P not divisible by p is $\{(p-1)!\}^{m/p}$ and in $P - 1$, the term

$$\{(p-1)!\}^{m/p} - 1 = [(p-1)! - 1]\,[(p-1!)^{m/p-1} + \ldots +$$
$$\ldots + (p-1)! + 1] = [(p-1)! - 1]\,F(p).$$

$(p-1)! - 1 \not\equiv 0 \bmod p$, and the number of terms in $F(p)$ is odd and when they are combined in consecutive pairs from the end or

beginning, we see that F (p) is also not divisible by p. In fact

$$\{(p-1)!\}^{p^{\mu-1}} = hp^{\mu} - 1.$$

$\therefore$ $\quad P - 1 \not\equiv 0 \bmod m.$

Hence it follows that $P + 1 \equiv 0 \bmod p^{\mu}$. Thus the product of numbers prime to and less than p^{μ} is congruent to $-1 \bmod p^{\mu}$.

(b) Let $m = 2p^{\mu}$ where p is a prime $\geqslant 3$ and $\mu \geqslant 1$, and $\phi(2p^{\mu}) = n$. The integers prime to and less than m are

$$\left.\begin{array}{l} rp + 1, \; rp + 3, \; \ldots rp + p - 2 \\ tp + 2, \; tp + 4, \; \ldots tp + p - 1 \end{array}\right\} \quad \ldots \quad (3)$$

where r is 0 or runs through even values from 2 to $2(p^{\mu-1} - 1)$ and t runs through odd values from 1 to $2p^{\mu-1} - 1$. The numbers in (3) are a complete set of roots of $x^n - 1 \equiv 0 \bmod m$. If γ be a primitive root of m, then $\gamma, \gamma^2, \ldots \gamma^n$ are also all the incongruent roots of the congruence. Denoting the numbers in (3) by $\alpha_1, \alpha_2, \ldots \alpha_n$ and their product by P, we have as before $P - 1 \equiv 0$ or $P + 1 \equiv 0 \bmod m$. Either is congruent to 0 mod 2.

The only term in P not divisible by p is $(p-1)!^{p^{\mu-1}}$.

As before, $\qquad P + 1 \equiv 0 \bmod 2p^{\mu}.$

(c) Let $m = 2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_{\mu}^{\alpha_{\mu}}$ where p's are odd primes, and $\phi(m) = n$. $n \equiv 0 \bmod 4$. The integers prime to and less than m are

$$\beta_1, \beta_2, \ldots \beta_n \quad \ldots \quad (4)$$

The primitive λ-roots of m can be divided into $N/p[\lambda_p(m)]$ sets, and the product of the primitive λ-roots of any set and hence the product of all the primitive λ-roots of m is congruent to 1 mod m, provided that $\lambda(m) > 2$. If N be the number of the primitive λ-roots of m, $N \equiv 0 \bmod 4$. The primitive λ-roots of m are some numbers of (4). The remaining numbers of (4) fall in pairs such that the product of the numbers in each pair is congruent to $-$ mod m. See (C).

For if a be any one of the remaining numbers, the congruence $ax \equiv -1 \mod m$ gives $x = -a^{\lambda(m)-1} = -\mu m + a'$, *i.e.*, $x = a' < m$, a' is also one of the remaining numbers. If not, it must be a primitive λ-root of m, *i.e.*, $a', a'^2, a'^3, \ldots a'^{\lambda(m)}$ will, when divided by m, leave different remainders prime to and less than m, *i.e.*, $\{\mu m - a^{\lambda(m)-1}\}^r$ $r = 1, 2, 3, \ldots \lambda(m)$ will, when divided by m, leave different remainders. Hence

$(-)^r a^{r\lambda(m)-r} = (-)^r \{(a^{(r-1)\lambda(m)} - 1)\, a^{\lambda(m)-r} + a^{\lambda(m)-r}\}$,

i.e., $(-)^r a^{\lambda(m)-r}$ will leave different remainders.

Hence a must also be a primitive λ-root which is not so. It follows that the product of all the remaining numbers is congruent to 1 mod m. Thus if P is the product of numbers prime to and less than m $P \equiv 1 \mod m$.

NOTE. Remembering that there exist two numbers γ, γ' [other than primitive roots, in general, of p^α (or $2p^\alpha$] prime to and less than p^α (or $2p^\alpha$) when p is a prime of the form $4n+1$, such that

$$\gamma\gamma' \equiv 1 \mod p^\alpha \text{ or } 2p^\alpha,$$

the cases (a) and (b) also can be similarly proved.

GLEANINGS.

33. Before Fermat, Vieta occupied important positions in the state administration. In more recent times, still in France, we find Bailly and Condorcet, two victims of the French revolution and a little later, Monge and Carnot who played important roles in the same period If Laplace showed for public affairs absolutely negative qualities, Fourier occupied with general satisfaction, and for a long time, the important post of Prefect. In the biography of Jacobi one has recently discovered a parenthesis of a political character in 1848. Finally if Ch. de Freycinet did not make a considerable mark in our science, it is because he had better developed his talent for public life. One learns of this from the splendid discourse of welcome of M Picard, who succeeded him at the Academie Francaise in 1928.

I shall conclude by remarking that in past decades, the miserable political conditions of my own country did not make it possible for Italian mathematicians to participate in public life; but in more recent times one meets with G. B Giorgini (friend of Chasles), F. Brioschi, E. Betti, L. Cremona, and U. Dini, who were successful in handling questions relative to the administration of the state.

G LORIA, in *Scripta Mathematica*, Vol. I, March 1933.

A NOTE ON JULIA'S LEMMA.

By K. Venkatachaliengar, *Institute of Science, Bangalore.*

The following formulation of Julia's Lemma* is easy to prove and appears to be interesting. What is proved here is not the sharpest form of the Lemma as given by Caratheodory†, but this formulation is given at the end of the paper. The proof of this can be easily constructed along the lines of Nevanlinna's proof given in his remarkable paper on bounded functions‡. Many of the results contained there can be proved more easily by formulating the results in the new way.

The new formulation of Julia's Lemma is this

(A) *Let* $w = f(z)$ *be defined and analytic in* $R(z) > 0$, *and let* $R(w) \geqslant 0$. *Let* $z_1, z_2, z_3, \ldots, z_n, \ldots$ *be a sequence of points on the positive part of the real axis such that* $z_n \to \infty$ *monotónically, and let* $R(w_n) \geqslant R(z_n)$.

Then $R(w) \geqslant R(z)$ *for all* z, *the equality sign occurring only when* $w = z + \alpha i$, *where* α *is a real constant.*

Proof. Let z'_n and w'_n be the images of z_n and w_n in the imaginary axis. Then $\phi(t) = (w - w_n)/(w - w'_n)$, considered as a function of $t = (z - z_n)/(z - z'_n)$ is regular and $|\phi(t)| < 1$ for $|t| < 1$. Moreover $\phi(0) = 0$. Hence by Schwarz's Lemma $|\phi(t)| \leqslant |t|$ in $|t| < 1$. Now the circle $|t| < 1$ corresponds to $R(z) > 0$. Hence

$$\text{(B)} \qquad \left| \frac{w - w_n}{w - w'_n} \right| \leqslant \left| \frac{z - z_n}{z - z'_n} \right| \text{ if } R(z) > 0.$$

Now, if (A) is not true, there is a point z' such that $R(w') < R(z')$, where $w' = f(z')$. By using (B) we will obtain an absurdity.

Let $k_n = |(z' - z_n)/(z' - z'_n)|$, and O be the origin. Let P be the nearer intersection of the circle $\left| \frac{z - z_n}{z - z'_n} \right| = k_n$ with the

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* Bieberbach : *Lehrbuch der Funktionen theorie* Bd. II, p. 112.

† *Loc. cit.*

‡ *Fest schrift an Lindlof* (Helsingfors, 1930).

real axis. Then $OP = z_n (1 - k_n) / (1 + k_n)$. From (B) we deduce that for all z which belong to the circle C_1 given by $\left| \dfrac{z - z_n}{z - z'_n} \right| \leqslant k_n$, the corresponding values of w are contained in the circle C_2 given by

$$\left| \frac{w - w_n}{w - w'_n} \right| \leqslant k_n.$$

Let the line $w_n w'_n$ cut the imaginary axis in O′ and the circle C_2 in two points of which the nearer is P′. Then

$$O'P' = R(u_n) . \frac{1 - k_n}{1 + k_n} = \frac{R(w_n)}{z_u} OP \geqslant OP.$$

Now the real part of w in C_2 is at least equal to O′P′. Therefore we obtain that the real part of w' is at least equal to OP. Now from elementary geometry it is clear that as $z_n \to \infty$, $OP / R(z') \to 1$. Hence we deduce that the real part of w' is at least equal to the real part of z'. This contradicts our preceding hypothesis. Hence we obtain that $R(w) \geqslant R(z)$ always. Now suppose the equality sign occurs for some point in the half plane $R(z) > 0$. Now consider the function $w - z$. This is regular in $R(z) > 0$, and we have $R(w - z) \geqslant 0$. Now by the maximum-modulus principle, we obtain that the supposition is only true if $w - z = \alpha i$. This completes the proof. The formulation of the Lemma in the sharpest known form is as follows:

With the above notation, let there be a sequence z_n, such that $z_n \to \infty$ and $w_n \to \infty$. Suppose

$$\lim_{n \to \infty} \left| \frac{w_n}{z_n} \right|^2 \frac{R(z_n)}{R(w_n)} = k, \text{ a finite constant.}$$

Then there exists a number $\lambda \geqslant k$, such that in any region in which $I(z) / R(z)$ is bounded, w/z, $R(w)/R(z)$, and $w'(z)$ tend to the same limit λ as $z \to \infty$. [By supposition $R(z) \to \infty$].

ON THE ORTHO POINT-LINE TRANSFORMATION.

By K. Rangaswami,

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1. In a brief discussion "On conics having a common self-polar triangle" Mr. K. Satyanarayana* has defined the ortho-line of a point P with respect to a triangle ABC as the line of collinearity, p, of the points L, M, N in which the lines through P perpendicular to PA, PB, PC meet the sides of the triangle ABC. That a line p has two ortho-points of which p is the ortho-line—is also evident; for it may be any one of the points in which the circles on AL, BM, CN as diametres meet. The purpose of this note is to draw attention to an alternate definition of the ortho-line of a point and to discuss synthetically a few interesting properties connected with the ortho-line.

2. Let S be the unique conic having ABC for a self-polar triangle and P for a focus. Since PA, PL are perpendicular lines through the focus, the pole of PA lies on PL; but it lies also on BC and is hence the point L. Similar considerations apply to the lines PB and PC. Hence L, M, N are collinear and the line of collinearity, p, is the directrix corresponding to P of S. Thus:

> The ortho-line p of a point P with respect to a triangle ABC is the directrix corresponding to P of the unique conic having ABC for a self-polar triangle and P for a focus ... (2.1)

Consider the pencil of conics through A, B, C, P and let H be the unique rectangular hyperbola in the pencil. By reciprocating with respect to a circle with centre at P and denoting the reciprocal elements by corresponding small letters, we see that the Fregier point of P for a conic Σ through A, B, C, P will reciprocate into the directrix of a parabola σ touching the lines a, b, c. Since these directrices pass through the ortho-centre of the triangle formed by the lines a, b, c that is, the centre of the unique circle s having abc for a self-polar

* *Vide The Math. Student*, Vol I, No. 1 March (1933), pp. 19—24.

triangle, the locus of Fregier points of P for all conics through A, B, C, P is the polar of P with respect to S. Hence

The ortho-line of P with respect to a triangle ABC is the locus of Fregier points of P for all conics through A, B, C, P ... (2.2)

In particular, considering the rectangular hyperbola H, since any chord of H subtending a right angle at P must be perpendicular to the tangent at P to H, and therefore parallel to the normal at P, the Fregier point of P for H is the point at infinity on the normal at P to H. Thus:

The ortho-line of P is parallel to the normal at P to the unique rectangular hyperbola through A, B, C, P ... (2.3)

If D is the ortho-centre of ABC, the rectangular hyperbola H is the same for the four triangles formed out of A, B, C, D. Further, H passes through the ortho-centres O_{23} of PBC, O_{14} of PAD, etc. Since AD is perpendicular to BC, the chord $O_{23}O_{14}$ of H subtends a right angle at P and is, therefore, parallel to the normal at P to H. Thus:

The ortho-lines of P with respect to the four triangles formed out of an ortho-centric tetrad ABCD and the joins of the ortho-centres of pairs of triangles like PBC, PAD, etc., are all parallel ... (2.4)

Let PT be the tangent at P to H, and p_∞ the point at infinity on p. Now $(L, M, N, p_\infty) = P(L, M, N, p_\infty) = P(A, B, C, T) = D(A, B, C, P)$ (by Chasles' theorem) is a constant if P moves on a line through D. Hence:

The ortho-lines of the points of a line through the ortho-centre of the triangle ABC, envelope a parabola inscribed in the triangle ... (2.5)

The points L, M, N are the Fregier points of P for the three line pairs in the pencil of conics through A, B, C, P. Since p is parallel to normal at P to H, the axis of S through P is the tangent at P to H. Further, p meets H in a pair of points which subtend a right angle at P.

3. Now, the ortho-points P, P′ of a line p are the two points of intersection of the circles on AL, BM, CN as diameters. If Γ is the polar circle of ABC, Γ is out-polar to all conics touching p and the sides of ABC and hence cuts all the members of the co-axal system

formed by their director circles orthogonally. Hence Γ belongs to the conjugate co-axal system and, therefore, has the limiting points P, P′ of this system for inverse points. Thus:

The two ortho-points of a line p with respect to a triangle ABC are inverse points with respect to the polar circle of ABC ... (3.1)

If H′ is the unique rectangular hyperbola through A, B, C, P′, the normals at P, P′ to H, H′ are both parallel to p. Hence:

If P and P′ are inverse points with respect to the polar circle of a triangle ABC, the tangents at P, P′ to the rectangular hyperbolas through A, B, C, P; A, B, C, P′ are parallel ... (3.2)

Let S′ be the unique conic through A, B, C, P, P′. Since S′ is out-polar to Γ, the ϕ-conic of S′ and Γ is the reciprocal of S′ with respect to Γ. But since P, P′ are inverse points with respect to Γ, PP′ must touch the ϕ-conic. Hence, the pole of PP′ with respect to Γ, *viz.*, the point at infinity, P_∞ in the direction perpendicular to PP′ must lie on S′. Also if p meets PP′ in P_0, $(LMNP_0) = P'(LMNP_0)$ $= P'(ABCP_\infty)$ on the conic S′ = the cross ratio of the points in which PP_∞ meets BC, CA, AB, PP′, all these lines being tangents to the ϕ-conic of S′ and Γ. Thus:

If P, P′ are inverse points with respect to the polar circle of a triangle ABC, the ortho-line of P (or P′) touches the ϕ-conics of the polar circle and the unique conic through A, B, C, P, P′ ... (3.3)

Now the chord $P'P_\infty$ of S′ subtends a right angle at P. Hence $P'P_\infty$ meets p in the Fregier point Q of P for S′. Further if PP′ meets BC, CA, AB in A′, B′, C′ respectively,

$$(LMNQ) = P_\infty(ABCP) = (A'B'C'P').$$

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34. God is that actuality in the world, in virtue of which there is physical law.

A. N. Whitehead: *Process and Reality.*

NOTES AND DISCUSSIONS.

The Editor welcomes for publication under this heading, brief discussions of interesting problems, critical comments, and suggestions likely to be helpful in the class-room.

Note on Euler's Constant.

The proof given in Chrystal's *Algebra* (p. 81, Part II, first edition) of the existence of this constant leaves something to be desired.

The problem is to show that, as $n \to \infty$, the function

$$S_n \equiv 1 + \frac{1}{2} + \dots + \frac{1}{n} - \log n \qquad (n = 1, 2, 3, \dots) \dots \quad (1)$$

tends to a limit γ such that $1 > \gamma > 0$.

In the proof given it is shown that S_n, for all values of n however great, lies between 0 and 1; and from that it is left to be understood that S_n tends to a limit γ lying between 0 and 1. This is hardly fair to the reader.

As a matter of fact S_n, starting from the value 1, decreases as n increases. The reader is not *helped* to realize this; but a reader, realizing this, might ask "could not the limit be zero itself?"

Consider, for example, the function $2^{1/n} - 1$, $(n = 1\ 2, 3, \dots)$. This also lies between 0 and 1 for values of n however great. But its limit is 0 and not a quantity lying between 0 and 1.

A very informing way of dealing with the point thus raised is to consider the companion function

$$T_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \log(n + 1), \quad (n = 1, 2, 3, \dots) \dots \quad (2)$$

Both S_n and T_n can now be treated by means of the inequality used in the proof, which may be written

$$\frac{1}{n+1} < \log(n + 1) - \log n < \frac{1}{n} \qquad \dots \quad (3)$$

It can be seen from it

(1) that S_n (having the starting value 1) decreases as n increases;

(2) that T_n has the starting value $1 - \log 2$, which is *positive*;

(3) that T_n increases as n increases;

and (4) that $S_n - T_n$ is positive, but tends to 0 as $n \to \infty$.

Hence, it follows that both S_n and T_n tend to a common limit γ such that

$$1 > \gamma > 1 - \log 2 > 0 \qquad \ldots \quad (4)$$

It is also seen that *every* S_n is greater than *every* T_n, and that γ lies *in* between them.

V. Ramaswami Aiyar.

On the Equation to a pair of tangents to a conic.

1. The following is a simple method for obtaining the combined equation to the tangents from a point $P(x_1, y_1)$ to a conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

in the form $SS_1 = T^2$, and compares favourably with the usual text-book methods in point of elegance ... (1)

2. Lemma. If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points through which a pair of parallel lines are drawn cutting (1) in P_1P_2; Q_1Q_2 then

$$\frac{PP_1 \cdot PP_2}{QQ_1 \cdot QQ_2} = \frac{S_1}{S_2} \qquad \ldots \quad (2)$$

For if θ is the angle made by these parallel lines with the x-axis, PP_1, PP_2, are the roots of the equation

$$r^2 (a \cos^2 \theta + 2h \cos \theta \sin \theta + b \sin^2 \theta)$$

$$+ 2r\, [(ax_1 + hy_1 + g) \cos \theta + (hx_1 - by_1 + t) \sin \theta] + S_1 = 0$$

From this and the similar equation for QQ_1, QQ_2 (2) follows.

3. Now let Q be any point on either tangent from P to (1). Let, T be the point of contact of this tangent. Then taking the line PQ, we get from (2)

$$\frac{PT^2}{QT^2} = \frac{S_1}{S_2} \qquad \dots \quad (3)$$

But the ratio PT/QT is equal to the ratio of the distances of P and Q from the chord of contact of the tangent, from P, *viz.*,

$T \equiv axx_1 + h\,(xy_1 + x_1y) + byy_1 + g\,(x + x_1) + f\,(y + y_1) + c = 0.$

Hence $$\frac{PT^2}{QT^2} = \frac{T_1^2}{T_2^2} = \frac{S_1^2}{T_2^2} \qquad \dots \quad (4)$$

Equating (3) and (4) and changing (x_2, y_2) into (x, y)

we get $$\frac{S_1}{S} = \frac{S_1^2}{T^2} \quad \text{or} \quad SS_1 = T^2.$$

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On the Multiplication of certain numbers.

Anent Mr. K. Subba Rao's note on the "*Multiplication of numbers ending in 5,*" (*Math. Student,* Vol. II, No. 2, June 1934), the following will be read with interest.

1. The product of two numbers whose last digits total up to 10, but whose other digits are identical, *i.e.*, numbers of the type ax and, ay, where $x+y=10$, is given by $[(a+1)\times a]p$ where p stands for the actual product $x \times y$.

Thus (i) $194 \times 196 = (20 \times 19)\ 24 = 98024,$

(ii) $85 \times 85 = (9 \times 8)\ 25 = 7225.$

2. The special case, where $x=y=5$, [example (ii) above], gives a rule for writing down the square of a number ending in 5, *viz.*, $(a5) \times (a5) = [(a+1) \times a]\ 25$. Thus $75^2 = 5625$ and $155^2 = 24025$.

3. An extension of the above case, where each number ends in 5, but the other figures are different, is the one given by Mr. Subba Rao;

and the result may be expressed in the following *somewhat* simpler manner.

$$(a5) \times (b5) = \left[a \times b + \frac{a+b}{2} \right] 25,$$

where, should the proper fraction $\frac{1}{2}$ occur while simplifying the terms in the rectangular brackets, it has the effect of increasing the digit in the tens' place by 5, (being in fact equal to half a hundred); *i.e.*, changing 25 to 75.

Thus $$45 \times 95 = \left[4 \times 9 + \frac{4+9}{2} \right] 25 = 4275 \quad \text{and}$$

$$85 \times 155 = \left[8 \times 15 + \frac{8+15}{2} \right] 25 = 13175.$$

These rules are useful when we have to deal with numbers of two or three digits; but they lose much of their utility when applied to numbers having more digits.

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86. **Mathematics and Law.** What men have achieved something of note in each of the fields mathematics and law? Five names occur to one rather readily. The great French mathematician Prine de Pierre (c 1608—1665) was councillor for the parliament of Toulouse and became noted for his knowledge of law. The versatile and able German mathematician and philosopher Gottfried Wilhelm Freiherr von Leibnitz (1646—1716) spent some time in diplomatic service and published material for a code of international law The Englishman John Wilson (1741—1793) who was the senior wrangler of 1761. a good teacher of Mathematics, and the discoverer of Wilson's theorem, became a Judge of common pleas (1786—93), and in our own day two more Frenchmen are to be mentioned, Painleve and Borel. Paul Painleve (b. 1863) professor of rational mechanics? .. .. President of the Chamber of Deputies in 1906, minister of public instruction and inventions in the Briand Cabinet (1915-1916), minister of war in the Ribot Cabinet (1917), and also premier for part of the same year in 1924, President of the Chamber of Deputies, in 1925 again Premier, and in 1926—28 minister of war in two successive cabinets formed by Poincare.

Emile Borel (b. 1871). Professor of the Calculus of Probability and of Mathematical Physics at the Sorbonne, was elected to the Chamber of Deputies in 1924 and was minister of marine in Painleve's cabinet in 1925,

R. C. ARCHIBALD. in *Scripta Mathematica*, Vol. I, December 1932.

Question 1635.

(K. J. SANJANA) :—(i) If n distinct primes of the form $4m + 1$ are multiplied together, show that the product can be resolved into a sum of two squares in 2^{n-1} ways.

(ii) Prove that the nth power of a prime of the form $4m + 1$ can be represented as a sum of two squares in $\frac{1}{2}(n + 1)$ or $\frac{1}{2}n$ ways according as n is an odd or even integer.

Solution by S. Minakshisundaram.

The proofs of the two theorems depend on the fact that any prime of the form $4m + 1$ can be uniquely expressed as the sum of two squares. (See Reid: *Theory of Algebraic Numbers*, p. 171, Art. 9).

(i) Let $p_1, p_2, \dots p_n$ be the n distinct primes of the form $4m + 1$ and $p_r = a^2_r + b^2_r$. It is evident then that the product $p_1 p_2 \dots p_n = \prod\limits_{r=1}^{n} (a_r^2 + b_r^2)$ can be expressed as the product of two conjugate complex numbers in 2^{n-1} ways.

(ii) Again if $p = a^2 + b^2$ then $p^n = (a^2 + b^2)^n$, and

$$p^n = \{(a + bi)^{n-r} (a - bi)^r\} \{(a + bi)^r (a - bi)^{n-r}\}.$$

(*a*) If n is an even integer, r can be allowed to take all values from 0 to $\frac{1}{2}n - 1$. When $r = \frac{1}{2}n$

$$p^n = \{(a+bi)^{n/2}(a-bi)^{n/2}\} \{(a+bi)^{n/2}(a-bi)^{n/2}\} = (a+bi)^n (a-bi)^n$$

the same as when $r = 0$; and when $r > n/2$ we will obtain one or other of the forms already found. Thus there are $n/2$ ways of expressing p^n as the sum of two squares.

(*b*) If n is an odd number r can take all values from 0 to $\frac{1}{2}(n-1)$ and when $r > \frac{1}{2}(n - 1)$ we will obtain one or other of the forms already obtained. Hence there are $\frac{1}{2}(n - 1) + 1$ or $\frac{1}{2}(n + 1)$ ways of expressing p^n as the sum of two squares.

A more general result is :—

Let n be a composite number of the form

$$2^{n}\prod_{r}(4n_r+1)^{\alpha_r}\prod_{s}(4n_s-1)^{\beta_s}$$

where $(4n_r+1)$ and $(4n_s-1)$ are primes.

(i) A necessary and sufficient condition that n may be expressed as the sum of two squares is that all the β's are even.

(ii) If the condition is satisfied, the number of ways in which n can be represented as the sum of two squares is

$$\tfrac{1}{2}\prod_{r}(\alpha_r+1) \text{ or } \tfrac{1}{2}\prod_{r}(\alpha_r+1)+\tfrac{1}{2}$$

according as some or more of the α's are odd; (Reid: *Algebraic Numbers*, p. 191 art. 16).

Questions 1641 & 1644.

(C. N. SRINIVASIENGAR) :—A plane curve of degree n has δ nodes, k_1 keratoid cusps, k_2 tacnodes, and k_3 ramphoid cusps. Prove that

(i) $m = n(n-1)-2\delta-3k_1-4k_2-5k_3$.

(ii) $n = m(m-1)-2\tau-3i-4k_2-5k_3$.

(iii) $i = 3n(n-2)-6\delta-8k_1-12k_2-15k_3$.

(iv) $i-k_1 = 3(m-n)$.

Solution and remarks by the Proposer.

1. Suppose a plane curve has a cusp of any kind. Taking the origin at the cusp and $y=0$ as the cuspidal tangent, the cartesian equation of the curve is

$$y^2=(a_0x^3+3a_1x^2y+3a_2xy^2+a_3y^3)+(b_0x^4+4b_1x^3y+\dots)+\dots$$

The necessary and sufficient conditions* for a ramphoid cusp at O are (i) $a_0=0$. (ii) $y^2-3a_1x^2y-b_0x^4=$ a perfect square, a_1 and b_0 being not zero. (iii) $27a_2a_1^2-24a_1b_1+4c_0\neq 0$ Condition (i)

* Use Newton's diagram, *vide* Hilton : *Plane Algebraic Curves*.

without (ii) is the necessary and sufficient condition for a tacnode at O. When $a_0 \neq 0$, the origin is an 'ordinary' or a 'keratoid' cusp.

If O be a tacnode, the expansion of the curve near O consists of two *linear* branches of the form $y = Ax^2 + Bx^3 + \dots$ But if O be a ramphoid cusp, the expansion near O consists of a super-linear (cyclical) branch of order 2, wherein the lowest term in x begins with x^2. It may be written $y = bx^2 \pm cx^{5/2} + dx^3 + \dots$ The expansion for an ordinary keratoid cusp is of the form

$$p = \pm ax^{3/2} + bx^2 \pm cx^{5/2} + dx^3 + \dots$$

It is also known that the reciprocals of a keratoid cusp, a tacnode and a ramphoid cusp are respectively tangents at an inflexion, a tacnode, and a ramphoid cusp (*vide* Hilton *loc cit.*, Chapter VI, § 5).

2. Plucker's equations, as they are usually written assume that the nodes and cusps are ordinary ones. When a curve possesses a tacnode, a ramphoid cusp, or other higher singularities, Plucker's equations can be retained without being blown up only by a very artificial process, which consists in supposing that any such multiple point O is equivalent to a particular combination of δ ordinary nodes κ ordinary cusps, τ bitangents with coincident points of contact at O, i inflexional tangents with inflexions at O. The numbers δ, κ, τ, i can be determined in each concrete instance by the help of the generalised definition for deficiency, but there is no guarantee that they are all positive integers.

3. Let us start with the general equation

$$y^2z^{n-2} + (a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3)z^{n-3}$$
$$+ (b_0x^4 + 4b_1x^3y + 6b_2x^2y^2 - 4b_3xy^3 + b_4y^4)z^{n-4} + \dots$$

where $z = 1$. Unless a_0 and a_1 are both zero, the origin is an ordinary point on the first polar of any point and the tangent coincides with the cuspidal tangent. The expansion of the first polar at O is therefore of the form

$$y^2 = A_0x^2 + A_1x^3 + \dots \dots$$

$2A_0$ gives the value of d^2y/dx^2 at (0, 0) for the first polar. The value of d^2y/dx^2 at (0, 0) for the given curve is $-3a_1$ when $a_0 = 0$. It will

be found that $-3a_1 = 2A_0$ if and only if $9a_1^2 = 4b_0$, that is, if O be a ramphoid cusp. Hence, if O be a ramphoid cusp, the coefficients of x^2 in the expansion of the first polar of any point and in that of the super-linear branch of the curve at O are equal. It follows that the number of intersections of the curve and a general first polar curve are four at a tacnode and five at a ramphoid cusp (*vide* Hilton : Chap. VI).

Hence, if a n-ic has δ nodes, k_1 Keraloid cusps, k_2 tacnodes, k_3 ramphoid cusps, we have

$$m = n(n-1) - 2\delta - 3k_1 - 4k_2 - 5k_3 \quad \ldots \quad (1)$$

The reciprocal curve therefore gives

$$n = m(m-1) - 2\tau - 3i - 4k_2 - 5k_3 \quad \ldots \quad (2)$$

Let us assume

$$i = 3n(n-2) - 6\delta - 8k_1 - \lambda k_2 - \mu k_3 \quad \ldots \quad (3)$$

so that

$$k_1 = 3m(m-2) - 6\tau - 8i - \lambda k_2 - \mu k_3 \quad \ldots \quad (4)$$

Consistency requires $\lambda - 24 = -\lambda$; $\mu - 30 = -\mu$.

Eliminate, say, τ and i from (2), (3), (4), and compare with (1).

$\therefore \quad \lambda = 12, \mu = 15.$

The last equation $i - k_1 = 3(m - n)$ follows mechanically.

4. The equations (3) and (4) can be worked out directly by consideration of the Hessian. The work is a bit tedious, but it leads to an interesting result.

Unless $a_1 = 0$, a tacnode or a ramphoid cusp of the given curve will be a triple point on the Hessian, at which there are *three coincident tangents*. The co-efficients of x^4, x^3y, and x^5 are all absent in the equation of the Hessian when $a_0 = 0$, $z = 1$. Hence an inspection of Newton's diagram will indicate that the approximation to the curve in the neighbourhood of the origin is

$$y^3 + Ax^2y^2 + Bx^4y + Cx^6 = 0$$

C does not vanish when $a_0 = 0$, unless a_1 and b_0 are also zero. For the tacnode, therefore, the expansion of the Hessian at the origin consists of three linear branches of the form $y = \alpha x^2 + \beta x^3 + \ldots$, but

in special cases, there may be one linear branch and a **super-linear** branch of order 2. The expression $y^3 + Ax^2y^2 + Bx^4y + Cx^6$ becomes a perfect cube if and only if $4b_0 = 9a_1^2$, *i.e.*, if O is a ramphoid cusp. The expression is then

$$\left(y + \frac{3a_1}{2}x^2\right)^3.$$

Hence, for a ramphoid cusp, the expansion of the Hessian is

$$y = A_0x^2 + B_0\omega x^{\frac{7}{3}} + C_0\omega^2 x^{\frac{8}{3}} + \ldots \qquad (\omega^3 = 1)$$

where A_0 is also the coefficient of x^2 in the expansion of the given curve at O. It is easily verified that $B_0 = 0$.

It will now be evident that the curve and the Hessian intersect 12 times at a tacnode, and 15 times at a ramphoid cusp.

Hence follow equations (3) and (4). Moreover, since A_0 is the same for the three curves, we have the following theorem :

If a plane curve possesses a ramphoid cusp, the two branches of the curve at the cusp, the first polar of any point not on the cuspidal tangent, and the three partial branches of the Hessian at the cusp, all have the same curvature at the point.

This is the result in Question 1644.

5. A special case of a tacnode is when $a_0 = a_1 = 0$, ($b_0 \neq 0$). The tacnode will be a point of inflexion on the first polar of every point not on the cuspidal tangent, and it will be a quadruple point on the Hessian at which two tangents coincide with the cuspidal tangent. Plucker's equations, as modified above, remain unaffected.

[NOTE. Equation (1) has been deduced by Hilton from the theory of quadratic transformations. (*vide* Examples 3 and 4, p. 134, first edition). But his statement that a ramphoid cusp is equivalent to the combination of a node and an ordinary cusp is misleading, unless it is also accompanied by a certain number of inflexions and bitangents].

Question 1646.

(A. A. KRISHNASWAMY AYYANGAR) :—Show that the equation of the axis of the parabola $S = 0$ in rectangular cartesian co-ordinates can be put in either of the forms $\frac{\partial}{\partial x}(S_x^2 + S_y^2) = 0$ or $\frac{\partial}{\partial y}(S_x^2 + S_y^2) = 0$ where S_x and S_y stand for $\frac{\partial S}{\partial x}$ and $\frac{\partial S}{\partial y}$ respectively. Explain the geometrical significance of the above forms.

Solution by Hukham Chand.

Let $S \equiv (\alpha x + \beta y)^2 + 2gx + 2fy + c = 0$. We make this equation homogeneous by replacing x, y by x/z and y/z.

The point at infinity on S is $(\beta, -\alpha, 0)$ and the point at infinity on chords perpendicular to the axis is $(\alpha, \beta, 0)$. So the polars of these points, with respect to the parabola, and the polars of the circular points at infinity given by $S_x^2 + S_y^2 = 0$ form a harmonic pencil with vertex at $(\beta, -\alpha, 0)$. Thus the axis of the parabola may be obtained as the conjugate with respect to $S_x^2 + S_y^2 = 0$ of the line at infinity, and hence also as the polar of any point on the line at infinity. In particular taking the point at $(1, 0, 0)$ or $(0, 1, 0)$, we have the equations

$$\frac{\partial}{\partial x}(S_x^2 + S_y^2) = 0 \text{ and } \frac{\partial}{\partial y}(S_x^2 + S_y^2) = 0.$$

The geometrical significance of the above forms are brought out in the above treatment.

Also solved by S. P. Ranganathachar and A. L. Shaikh and N. K. Narasimhamurti.

Question 1653.

(K. J. SANJANA AND K. F. VAKIL):—In the triangle ABC, each of the angles at B and C is double of the angle at A ; CM is drawn perpendicular to AB ; AM and MB are bisected at F and G respectively and CF and CG are joined.

Prove that (1) $\sin FCG : \sin ACB = 8 : 11$;

(2) rect. AF.BG : rect. CF.CG $= 1 : 11$;

and (3) in-radius of ΔCFG : in-radius of ΔCAB $= 5 + \sqrt{5} : 10$.

Solution by P. C. Shah, A. L. Shaikh, B. M. Narayana Rao, K. Subba Rao, M. K. Hariharan and N. K. Narasimhamurti.

Let ABC be a triangle having $\angle B = \angle C = 2\angle A$.

$\therefore$ If AC = AB = y and BC = x, then $y^2 - x^2 = xy$;

and if CM $\perp$ AB, then AM $= (y+x)/2$ and BM $= (y-x)/2$.

$\therefore$ $AF.BG = \frac{1}{2} AM.\frac{1}{2} BM = (y^2 - x^2)/16 = xy/16$.

Now $CF^2 = AC^2 - 3AF^2$ from the ΔACM

$$= y^2 - 3(y+x)^2/16 = (13y^2 - 6xy - 3x^2)/16$$

$$= (y^2 + 6xy + 9x^2)/16 \text{ [substituting for } 12y^2]$$

$$= (y + 3x)^2/16.$$

$\therefore$ $CF = (y+3x)/4$.

Similarly $CG = (3y - x)/4$.

$\therefore$ $CF.CG = (3y^2 + 8xy - 3x^2)/16 = 11xy/16$. $\therefore \dfrac{AF.BG}{CF.CG} = \dfrac{1}{11}$

and $CF + CG + FG = (3y + x)/2$ since $FG = y/2$.

$$\therefore \quad \frac{\text{Perimeter of ABC}}{\text{Perimeter of FCG}} = \frac{2(2y+x)}{3y+x} = \frac{5+\sqrt{5}}{5}.$$

If r, r', s, s' and Δ, Δ', be the radii, semi-perimeter and areas of Δ FCG and Δ ABC, then

$$\frac{r}{r'} = \frac{\Delta}{s} \times \frac{s'}{\Delta'} = \frac{s'}{2s} = \frac{5+\sqrt{5}}{10} \quad \text{since } \frac{\Delta}{\Delta'} = \frac{1}{2}.$$

Again $$\frac{\Delta}{\Delta'} = \frac{\frac{1}{2} CF.CG \,.\, \sin FCG}{\frac{1}{2} AC.CB \,.\, \sin ACB} = \frac{11 \sin FCG}{16 \sin ACB} = \frac{1}{2}.$$

$$\therefore \quad \frac{\sin FCG}{\sin ACB} = \frac{8}{11}.$$

Question 1658.

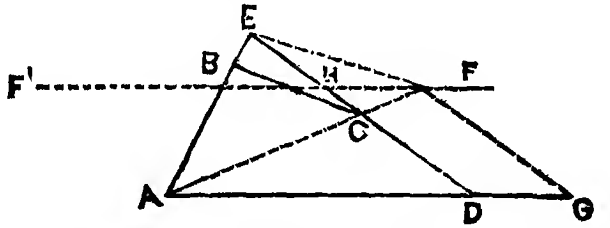
(A. RANGANATHA RAO) :—Construct a quadrilateral three of whose sides are equal, given, in magnitude, the remaining side and the angles adjacent to this side.

Solution by Nathan Altshiller-Court.

Let ABCD be the required quadrilateral, AD being the side given in magnitude. The triangle ADE, where E ≡ (AB, CD), is given by its base and the adjacent angles. If through E we draw the parallel to BC meeting AC in F and through F the parallel to ED meeting AD in G, we obtain the auxiliary quadrilateral AEFG which is homothetic to ABCD, A being the homothetic center; hence AE = EF = FG. Now the vertex F of the quadrilateral AEFG may be determined as follows. On DE lay off DH = AE, and the point F is common to the parallel through H to AD and the circle (E) having E for centre and EA for radius.

The line AF meets ED in the vertex C of the required quadrilateral ABCD, which quadrilateral is now readily completed.

The second point of intersection F′ of the line HF with the circle (E) yields a second solution of the problem. Neither of these two solutions, however, is necessarily a convex quadrilateral.



Bibliographical References. 1. I. Alexandroff: *Problemes de geometrie elementaire*, p. 72, Hermann, Paris, 1899. Trans. from the Russian; 2. Nathan Altshiller-Court: *College Geometry*, p. 44, Johnson Publishing Company, Richmond, Virginia; 3. *School Science and Mathematics*, 1929, p. 644, Q. 1062.

Also solved by R. S. Vaidiswaran, N. K. Narasimhamurti, S. Srinivasan and the proposer, and by trigonometric methods by V. Sundaram, M. K. Hariharan and K. Subramanian.

REVIEWS OF BOOKS.

A History of Mathematics in America before 1900, by DAVID EUGENE SMITH, Professor Emeritus of Mathematics, Teachers' College, Columbia University, and JEKUTHIEL GINSBURG, Professor of Mathematics in Yeshiva College, New York, and Editor of "*Scripta Mathematica.*" Chicago, Illinois, 1934, pp x + 209

This little volume was published by the Mathematical Association of America as number 5 of the "Carus Mathematical Monographs." It is practically confined to the history of mathematics in the United States of America and hence it relates to practically the same territory as the much larger work (400 pages) by F. Cajori which was published in 1890 by the Bureau of Education of the United States under the title *The Teaching and History of Mathematics in the United States.* While the title of the former work implies that it is restricted to American Mathematics *before* 1900, this restriction is not maintained and hence this title is somewhat misleading.

About one-half of the volume is devoted to the twenty-five years from 1875 to 1900, during which American mathematicians began to co-operate actively with those of Europe, especially with those of Germany, in the development of our subject. The earlier periods are almost completely barren as regards American contributions towards the development of mathematics but they are interesting from the standpoint of the teaching of this subject in our schools and Universities. In America nearly all of the advances in mathematics were made by the professors of this subject in the colleges and the Universities. The most notable exception to this rule is G. W. Hill (1838—1914) who was connected with the Nautical Almanac of the United States for many years (1859—1892) and was later President of the American Mathematical Society (1895-1896).

Although the English language is the predominant language in the United States of America, the mathematical developments in this country since the beginning of the nineteenth century have not been predominantly influenced by English writers. During the first half of this

century the works of French mathematicians were most assiduously studied in this country and served as models for many of the text-books which were then produced therein, while in the second half of the same century those of German authors gradually became more influential and the German Universities attracted most of the students of the United States of America who went to Europe for their advanced training in Mathematics. Since the World War, there has naturally been a great change in this respect but German mathematical literature is still very influential in this country as is evidenced by the translation into English of Felix Klein's *Elementary Mathematics from an advanced standpoint*, 1932.

The little volume under review is divided into four chapters. The third of these begins with the following sentence: "The first three centuries of our history were, as we have seen, barren of achievement in the domain of mathematics." It is somewhat difficult to harmonize this with the following quotation from page 10: "So it is not in the schools that the advance in mathematics in the sixteenth, seventeenth, and eighteenth centuries in America is to be found, but rather in the quasi-astronomical work of the astrologers and the makers of almanacs, the two classes often merging in one." On page 96 it is stated that John Farrar (1779—1853) "did much for elementary mathematics in this country through his translations 1818—1825) of the works of Euler, Lacroix, Legendre, and Bezout, and through his publication of a number of text-books."

In view of the enormous extent of these works the reader might at first be inclined to think that John Farrar was the most prolific mathematical writer that ever lived, especially since such strenuous efforts have been made in recent years to publish the collected works of L. Euler by international co-operation. These quotations are noted here because they may be interesting in themselves and may enable the reader to form a more nearly correct idea as regards the merits of the little volume under review than could otherwise be given in such a brief space. They seem to be characteristic examples of D. E. Smith's historical writings in recent years.

G. A. MILLER.

Elementary Quantum Mechanics: By GURNEY, Cambridge University Press, pp. 156. Price 8s. 6d.

The purpose of the book is to explain to the physicist not acquainted with the requisite mathematical foundations, the ideas of modern quantum dynamic theory. Apart from certain indispensable mathematical assumptiouns the author does not presuppose an acquaintance with higher mathematics. The great idea of Schodinger, *viz.*—"The correlation of the quantum problem in physics to an eigen-value problem in mathematics" has been explained in a nice way by means of the idea of "fitting ψ-patterns into potential boxes." This is made clear by means of a number of examples such as the linear harmonic oscillator and the hydrogen atom. It is of course impossible thereby to comprehend fully the relation of the eigen-value problem to the modern theory; but a rongh understanding of the theory and the consequent changes in the attitude of the physicist towards the allied physical events are certainly possible by studying the treatment found in the book. It would have been more valuable if the author had explained more fully how the shortcomings of the older theory have been overcome by means of these modern ideas.

The first three chapters of the book deal with the introduction of modern ideas and some simple problems. The fourth chapter deals with the uncertainty principle, the problem of many particles, the periodic table and the structure of metals. The fifth chapter is devoted to "the movement of particles" and ends with a list of select references. The later chapters of the book deal mainly with molecules. The author has attempted with a high measure of success to make the ideas clear to the non-mathematical scientist. There is also a chapter on the electronic theory of conductors and insulators from which one can obtain a rough idea of this important branch of the theory. The chapter on perturbation theory will be of considerable use to the physicist as the author has taken much pains to make the problem comprehendable without many mathematical assumptions. In the closing pages of the book descriptions of two physical events [atomic and molecular excitations and atomic collisions] are treated in detail. It is doubtful whether the mathematical appendix at the end of the book serves any useful purpose as the reader who likes to get acquainted with them can learn them much better by studying the classical treatises by Neumann Weyl, Dirac and Wan der Waaerden, etc.

Altogether, a very helpful book.

K. V. I.

ANNOUNCEMENTS AND NEWS.

Readers are invited to contribute to the value of this section by sending suitable news items of interest.

The following gentlemen have been elected members of the Society :—

Mahmad Ali Anwar, B.A. (Hon.), Dar ul-Aquil, Chahar Bagh, Jullunder

Ch. Sultan Baksh, M A., Government College, Hoshiarpur.

T. J Balwani, M.Sc , D. J. Sind College, Karachi.

Rabindranath Barna, Esq , M.Sc., B.L., Pleader, Jorhat, Assam.

G. S Daryanani, M Sc , S.T.C., Model High School, Karachi

S. P Kaushik, M Sc., Maharaja's College Bikaner

Dr. Victor Levin, Dr. Ing (Berlin). "The Retreat," Shahibag, Ahmedabad..

M. V. Vaidyanatha Sastri., Esq., M.A., Nizamiah Observatory, Begumpet (Deccan).

The Indian Physical Society which was recently formed with its head-quarters at Calcutta held its first annual meeting in January 1935. It has a membership of over 90 and besides arranging for lectures, symposiums, etc., publishes the ***Indian Journal of Physics.*** The office-bearers for 1935 are: President: Principal B. M. Sen, Calcutta; Vice-Presidents: Prof. J. B. Seth, Lahore, and Prof. Kanta Prasad, Patna; General Secretary: Prof D. M. Bose, Calcutta.

The twenty-second annual meeting of the Indian Science Congress was held at Calcutta from January 2nd to 8th, 1935, under the Presidentship of Dr. J. H. Hutton, Deputy Commissioner, Naga Hills, Assam. The sectional address for the Mathematics and Physics section was delivered by the sectional President, Dr. N. R. Sen, and dealt with "The Development of modern theoretical Physics and its limitations." The 1936 session of the Science Congres will, it is expected, meet at Indore under the Presidentship of Dr. Sir Upendranath Brahmachari

An international Congress for Problems of Vector and Tensor analysis was held at the University of Moscow in May 1934. The Congress was organised by Prof B. Kagan and Prof. J. A. Schouten.

In 1933 the University of Madras offered a Ramanujan Memorial Prize for the best original thesis by an Indian or one domiciled in India on a definite branch of Mathematics, pure or applied. A number of theses were submitted and the

University has now announced that the prize of the value of Rs. 900 has been divided equally among the following:

S. Chandrasekharan, Esq., Fellow, Trinity College, Cambridge;

S Chowla, Esq., Reade, Andhra University, Waltair;

D D. Kosambi, Esq., Professor of Mathematics, Fergusson College, Poona.

We are glad to announce that exchange relations have been recently established with the following journals:

Collected papers of the Faculty of Science, Series A, Osaka Imperial University, Osaka, Japan.

Journal of the London Mathematical Society.

Prof. C. E. Weatherburn of the University of Western Australia has been awarded the Hector Medal and Prize by the Royal Society of New Zealand, for his contributions to differential geometry.

It is with deep regret we record the sudden death in March 1925 of Dr. Ganesh Prasad, Hardinge Professor of Mathematics at the Calcutta University, President of the Calcutta Mathematical Society and Vice-President of the Benares Mathematical Society. Dr. Prasad attended a meeting of the Executive Council of the Agra University, and after making a short speech collapsed. He was removed to the Thomason Hospital where he died shortly after admission.

Dr. Prasad's connexion with the Calcutta University began in 1914 when, at the invitation of the then Vice-Chancellor, Sir Ashutosh Mookerjee, he relinquished the post of Professor of Mathematics of Queen's College, Benares, and accepted the chair of Applied Mathematics in the Calcutta University founded by Sir Rash Behari Ghosh.

The death is announced of Prof. W. Franz Meyer of the University of Knigsberg at the age of 77.

Prof. D. M Y. Somerville of Victoria College, Wellington, New Zealand, author of text-books on Analytical Geometry of 2 and of 3 dimensions and on Non-Euclidean Geometry, died in Jan. 1934 at the age of 54.

Madame Marie Curie, Professor of General Physics in the Paris University, and Director of the Laboratoire Curie at the Institute de Radium, known for her work with her husband Pierre Curie leading to the discovery of radium, died July 1934.

William de Sitter, Director of the Astronomical laboratory at Leiden and well-known for his cosmological speculations, died on Nov. 19th, 1934.

QUESTIONS FOR SOLUTION.

Proposers of Questions are requested to send their own Solutions along with their Questions.

1692 (K. RANGASWAMI):—Show that every pair of polar lines with respect to a quadric has another unique pair of polar lines intersecting the former pair, such that the two pairs meet the quadric in points the normals at which to the quadric belong to a linear complex.

Also, if the lines of the first pair are mutually perpendicular, the normals meet the line of shortest distance of this pair.

1693. (HANSRAJ GUPTA):—If P (n, m) denote the number of partitions of n in which the least part is m, and a is any positive integer not less than 3, prove that

$$\lim \frac{P(am-1, m)}{m^{a-3}} = \frac{1}{(a-3)!\,(a-2)!}.$$

1694. (V. THEBAULT)—France:—With the figures x, $x+1$, $x+2$, $x+3$, $x+4$ form a number whose square contains the figures 1, 2, 3, 4, 5, 6, 7, 8, 9 each taken once.

1695. (A. A. KRISHNASWAMY AYYANGAR):—"A coin is tossed 1000 times. Show that an absolute majority of the 2^{1000} possible sequences gives the difference between the number of heads and number of tails less than seven."

(Whittaker and Robinson: *The Calculus of observations*, p. 177).

Is this problem correct?

1696. (K. RANGASWAMI):—P, P′ are inverse points with respect to the polar circle of a triangle ABC. Show that the tangents at P, P′ to the rectangular hyperbolas through ABCP, ABCP′ respectively are parallel.

1697. (B. RAMAMURTI):—Given a conic and three points A_1, A_2, A_3 in the plane of the conic, show that in general there are two and only two triangles inscribed in the conic such that the sides taken in order pass through A_1, A_2 and A_3 respectively. Show that there are ∞' such triangles when and only when A_1, A_2, A_3 form the vertices of a triangle self-conjugate with respect to the conic.

Extend the result to n-gons inscribed in the conic such that the sides taken in order pass through n fixed points in the plane.

1698 (V. RAMASWAMI AIYAR) —Given four points A, B, C, D in a plane, let α, β, γ, δ be the isogonal conjugates of a point P at infinity with respect to the triangles BCD, CDA, DAB, ABC. Prove that

(i) The quadrangles $\alpha\beta\gamma\delta$, corresponding to different points P at infinity, are all similar and have a common centre of similitude

(ii) This centre of similitude is the "*Durairajan point,*" M, of the quadrangle ABCD and also of each of the quadrangle $\alpha\beta\gamma\delta$.

(iii) For any quadrangle $\alpha\beta\gamma\delta$ the rectangles MA.Mα, MB Mβ, MC.Mγ, MD.Mδ are all equal

[As to the point M, here, called the "Durairajan point," see Mr. Durairajan's question 1480 in our Journal; and also questions 1481, 1485 and 1495 by others].

1699. (A RANGANATHA RAO):—If O is a given point in the plane of a conic S, prove that

(i) there exist four points A, B, C, D on S such that the directrix of the parabola having a contact of the third order with S at each of these points, passes through O;

(ii) A, B, C, D lie on a circle whose centre is O.

(iii) the circle ABCD is out-polar to S;

(iv) (ABCD) is the unique cyclic tetrad on S with centre O, apolar to the feet of the normals to S from O.

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